Notes on Analytical Dynamics

Jan Peters & Michael Mistry

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1 Newtonian Mechanics

1.1 Basic Asssumptions and Newtons Laws

- Lonely pointmasses with positive mass.
- Newtons 1st: Constant velocity v in an inertial frame (i.e., no acceleration).
- Newtons 2nd: $F = d(mv)/dt$.
- Newtons 3rd: Action has reaction.
- 2 isolated particles: $m_1 v_1 + m_2 v_2 = p_\Sigma \Longrightarrow F_1 + F_2 = 0$.

1.2 Practical Pendulum Equations

We have a base acceleration $\mathbf{a}_0 = \ddot{x}_0\mathbf{i} + \ddot{y}_0\mathbf{j}$ and angular acceleration $\ddot{\theta}\mathbf{k} = \dot{\omega} \Longleftrightarrow \dot{\theta}\mathbf{k} = \omega$. This implies that

$$\mathbf{a}_B = \mathbf{a}_0 + \mathbf{a}_{B|0},$$

with

$$\mathbf{a}_{B|0} = \dot{\omega} \times \mathbf{R} + \omega \times (\omega \times \mathbf{R}).$$

1.3 Energy

We define

$$\Delta T = \int_{x_0}^{x_f} F(x)dx = \int_{x_0}^{x_f} -\nabla V(x)dx = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2,$$

where we use $T = mv^2/2 \geq 0$ as the kinetic energy, and have $F(x) = -\nabla_x V(x)$ for the potential energy for conservative systems. From this we also realize that $V(x_0) + T(x_0) = V(x_f) + T(x_f) = \text{const}$. Furthermore, we have $m\ddot{x} = F(x)$.

1.4 Langrangians from Energy

From Kinetic Energy, we can derive the generalized equations of motion. Assume we have $T = 0.5\sum_{i=1}^{n} m\dot{x}^2$ as kinetic energy for a system of particle, and the positions as functions of generalized coordinates, i.e., $x = x(q)$. In this case, we also have

$$\frac{\partial T}{\partial q_k} = \sum_i m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k} = \sum_i p_i \frac{\partial \dot{x}_i}{\partial q_k},$$
$$\frac{\partial T}{\partial \dot{q}_k} = \sum_i m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \sum_i p_i \frac{\partial x_i}{\partial q_k},$$

where the later part is only true for holomonic constraints (i.e., constraints do only depend on the generalized positions and time or, equivalently, $x_i = x_i(q_1, \ldots, q_n, t)$) for which we have "dot-cancellation". When differentiating the later of the two with respect to time, we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_i \dot{p}_i \frac{\partial x_i}{\partial q_k} + \sum_i p_i \frac{\partial \dot{x}_i}{\partial q_k}.$$

When adding up these equations, we realize that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \sum_i \dot{p}_i \frac{\partial x_i}{\partial q_k} = \sum_i F_i \frac{\partial x_i}{\partial q_k} = \tau_k,$$

where τ_i is generalized force. We can repeat the same excercise for any $V(q)$, where we obtain $\tau_k = -\partial V/dq_k$ as $\partial V/d\dot{q}_k = 0$. When defining the Lagrangian

$$L = T - V,$$

this implies that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0,$$

which is equivalent to saying that the force derived from V is equal to the one derived from T.

1.5 Duration of Motion

From

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x) = h,$$

we can infer the duration of movement for two points, i.e.,

$$T = \int_{t_0}^{t_f} dt = \sqrt{\frac{m}{2}} \int_{x_0}^{x_f} \frac{dx}{\sqrt{h - V(x)}}.$$

If $T = g(x_0, x_f, h)$ is invertable, we have $x = g^{-1}(x_0, t - t_0, h)$. Its usually intractable.

An additional representation is given by $V(q) = V_0(q)+\varepsilon V_1(q)$, which implies that $T(h) = T_0(h) + \varepsilon T_1(h)$ also has

$$
\begin{aligned}
T_0(h) &\approx 2\sqrt{2m}\frac{d}{dh}\int_{q_{10}}^{q_{20}} \sqrt{h - V_0(q)}dq,\\
T_1(h) &\approx -\sqrt{2m}\frac{d}{dh}\int_{q_{10}}^{q_{20}} \frac{1}{\sqrt{h - V_0(q)}}dq.
\end{aligned}
$$

1.6 Small Oscillations

When linearizing $0.5m\dot{x}^2 + V(x) = h_0$ around an equilibrium point x_0 with a small deviation Δx, we obtain $0.5m\Delta\dot{x}^2+V(x_0+\Delta x) = h_0+\Delta h$, and subtracting the two yields

$$
\frac{1}{2}m\Delta\dot{x}^2 + \frac{1}{2}\left.\frac{\partial^2 V}{\partial x^2}\right|_{x_0} \Delta x^2 = \Delta h.
$$

When comparing this equation to a linear oscillator, it yields the frequency and amplitude

$$
\begin{aligned}
\omega^2 &= \frac{1}{m}\left.\frac{\partial^2 V}{\partial x^2}\right|_{x_0},\\
A &= \sqrt{\frac{2\Delta h}{V''(x_0)}}.
\end{aligned}
$$

1.7 Phase planes

We can draw phase plane motions using the potential function. We have have the following rules, illustrated in Figure 1.

- Endpoints on the x-axis have $V(x) = h$. These are the amplitudes of the oscillation.
- Endpoints on the $\dot{x}$-axis have $T(\dot{x}, x) = h$.
- A hill is a seperatrix on the axis, a valley an attractor.

2 Hamiltonian Dynamics

2.1 Basic Idea

The basic idea of Hamiltonian Dynamics is to replace the variables x by the generalized coordinate q and $\dot{x}$ by the conjugate momentum p, where

$$
\begin{aligned}
x &= q,\\
p &= m\dot{x}.
\end{aligned}
$$

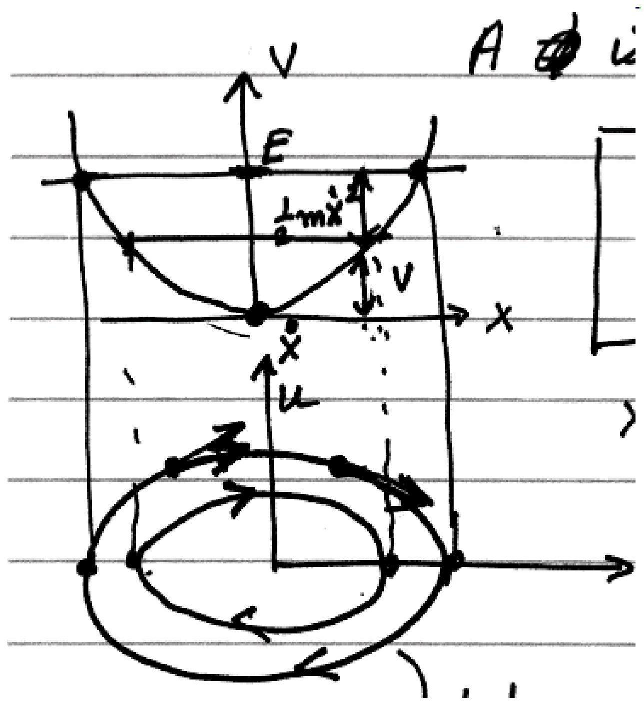

Figure 1: This figure demonstrates how to obtain a phaseplane motion from a potential function in a conservative system.

For example, for the Hamiltonian $H(q,p,t) = p^2/(2m) + V(q,t)$, where we see that

$$\dot{q} = \frac{p}{m} = \frac{\partial H(q,p,t)}{\partial p},$$
$$\dot{p} = F(q,t) = -\frac{\partial H(q,p,t)}{\partial q}.$$

When differentiating a time-invariant $H(q,p)$ with respect to time, we see that

$$\begin{aligned}\frac{dH(q,p)}{dt} &= \frac{\partial H(q,p)}{\partial q}\frac{dq}{dt} + \frac{\partial H(q,p)}{\partial p}\frac{dp}{dt}, \\ &= \frac{\partial H(q,p)}{\partial q}\frac{\partial H(q,p,t)}{\partial p} - \frac{\partial H(q,p)}{\partial p}\frac{\partial H(q,p,t)}{\partial q} = 0,\end{aligned}$$

i.e., that all trajectories have constant Hamiltonians or are on the isoclines.

2.2 Understanding Hamiltonians

A quantity is considered "conserved" if it is constant over time while the motion proceed, i.e., it is independent of the generalized velocities $\dot{q}_k$. For example, the energy $E = V + T$ is conserved. We can express this in general as the difference of the part of the Langrangian which depends explicitly on $\dot{q}_k$ and the whole langrangian, i.e.,

$$H \equiv \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L.$$

By differentiating, we obtain

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t},$$

which implies that H is only conserved if time does not appear explicitly in the Lagrangian. Additionally, we have

$$\frac{dH}{dq} = -\frac{\partial L}{\partial q}.$$

2.3 Area Preservation

When linearizing the resulting equations from the Hamiltonian, we obtain

$$q(t) = q(t_0 + \delta t) = q(t_0) + \left.\frac{dq}{dt}\right|_{t_0} \delta t + O(\delta t^2) = \underbrace{q(t_0) + \frac{\partial H}{\partial p}\delta t + O(\delta t^2)}_{f(q,p,t)},$$

$$p(t) = p(t_0 + \delta t) = p(t_0) + \left.\frac{dp}{dt}\right|_{t_0} \delta t + O(\delta t^2) = \underbrace{p(t_0) - \frac{\partial H}{\partial q}\delta t + O(\delta t^2)}_{g(q,p,t)}.$$

Now, we can also linearize dp and dq, and obtain

$$dp(t) = \frac{df}{dq}dq + \frac{df}{dp}dp = \left(1 + \frac{\partial^2 H}{\partial q \partial p}\right) dq + \left(\frac{\partial^2 H}{\partial p^2}\right) dp,$$

$$dq(t) = \frac{dg}{dq}dq + \frac{dg}{dp}dp = \left(-\frac{\partial^2 H}{\partial q^2}\right) dq + \left(1 - \frac{\partial^2 H}{\partial p \partial q}\right) dp.$$

When denoted as

$$\begin{bmatrix} dp(t) \\ dq(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 + \frac{\partial^2 H}{\partial q \partial p} & \frac{\partial^2 H}{\partial p^2} \\ -\frac{\partial^2 H}{\partial q^2} & 1 - \frac{\partial^2 H}{\partial p \partial q} \end{bmatrix}}_{A} \begin{bmatrix} dp(t_0) \\ dq(t_0) \end{bmatrix},$$

we realize that $\det A = 1 + O(\delta t^2) \approx 1$, i.e., that the area of any point in state space is constant and we have area preserving mapping. This area conservation is equivalent to Energy conservation, see Figure .

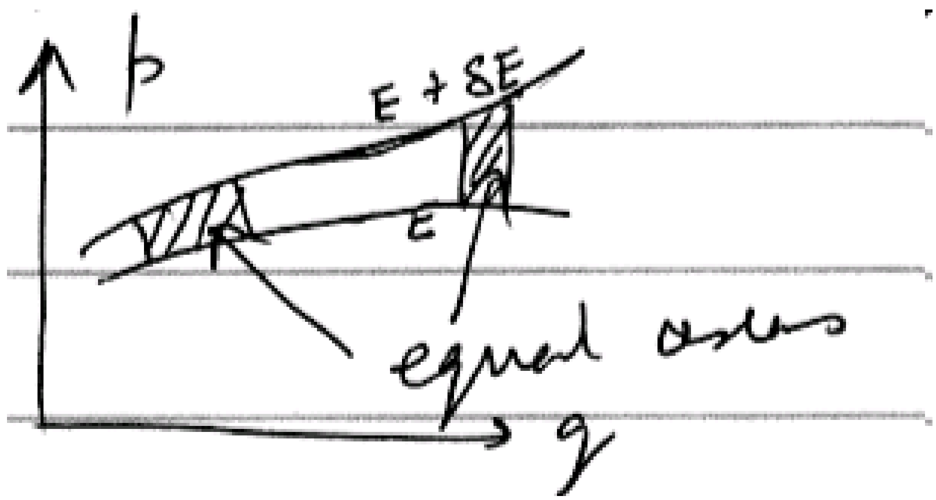

Figure 2: This figure illustrates the conservation of the area in (q, p) space which is equivalent to the conservation of energy.

2.4 Hamiltonians and Lagrangians

The connection between $\{x, \dot{x}\}$ and $\{q, p\}$ is not always a simple scaling operation but are in fact defined by some operations. We define the $\dot{q} = u$, and $p_k \equiv \partial L / \partial \dot{q}_k$. From

$$H \equiv \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = \sum_k u_k p_k - L.$$

This yields for a one particle system the equation

$$L\,(q, u, t) = pu - H(q, p, t).$$

We realize that $u = (L + H)/p$. This transformation is illustrated in Figure 3. Furthermore, we realize

$$\dot{q} = u = \frac{\partial H}{\partial p}.$$

The Lagrangian is defined as an function $L(q, u, t)$ in terms of position and the slope of the path with respect to q. The Hamiltonian is defined as a function $H(q, p, t)$ in terms of the position and the momentum. This is illustrated in Figure 4. If you are given a Hamiltonian such as $H(p) = p^2/(2m)$, you can always compute the Lagrangian which would be $L(u) = pu - H(p) = mu^2/2 = m\dot{q}^2/2$ as $u = \partial H / \partial p = p/m \Longleftrightarrow p = mu$ in this example.

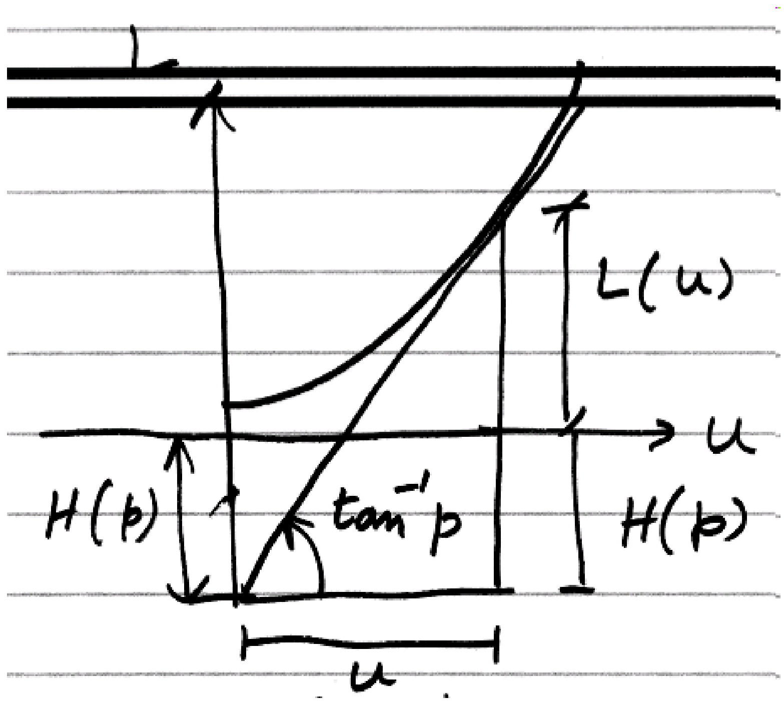

Figure 3: This Figure illustrates the transformation of a Lagrangian into a Hamiltonian.

3 Systems of Particles

3.1 Momentum

The motion of a system or particles can be described in terms of the motion $\mathbf{X}_c$ of the center of mass (i.e., the translation) and the rotation around the center of mass. If we define $\mathbf{y}_i = \mathbf{x}_i - \mathbf{X}_c$, and $M = \sum_i m_i$, we can obtain

$$\mathbf{L}_0 = M\mathbf{X}_c \times \dot{\mathbf{X}}_c + \sum_i m_i \mathbf{y}_i \times \dot{\mathbf{y}}_i = \sum_i m_i \mathbf{x}_i \times \dot{\mathbf{x}}_i,$$

where $\sum_i m_i \mathbf{x}_i = M\mathbf{X}_c$.

3.2 Torques

The torques around the the center of mass is given by

$$\tau_c = \sum_{i=1}^{n} \mathbf{y}_i \times \mathbf{F}_i = \frac{d}{dt}\left(\sum_i m_i \mathbf{y}_i \times \dot{\mathbf{y}}_i\right) = \dot{\mathbf{L}}_c,$$

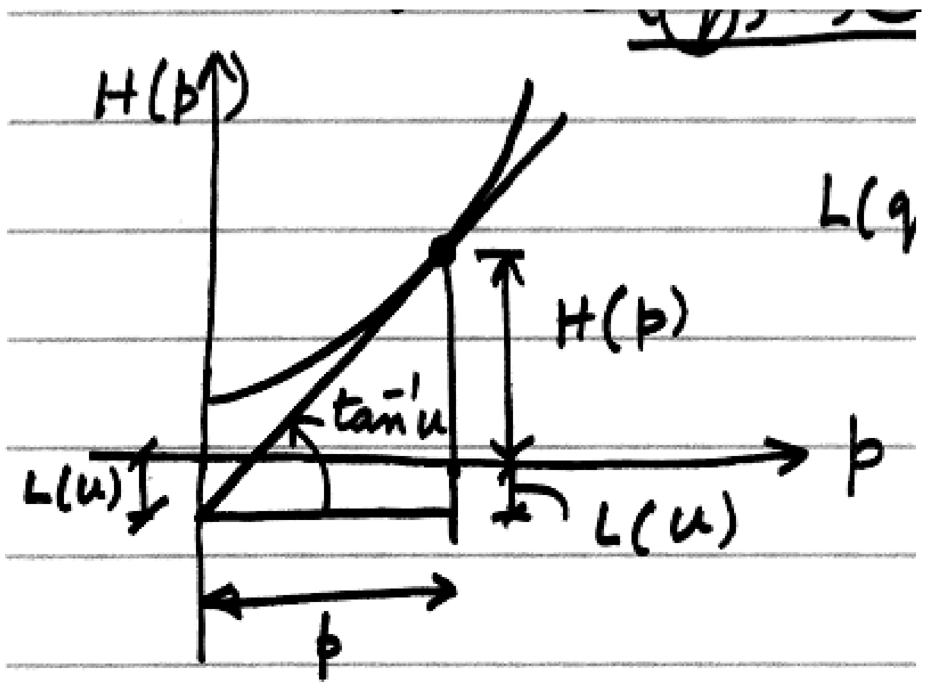

Figure 4: This figure illustrates the transformation of a Hamiltonian into a Lagrangian.

and by

$$\tau_{z_0} = \mathbf{r} \times M\ddot{\mathbf{X}}_c + \dot{L}_c,$$

where $M\ddot{\mathbf{X}}_c = \sum_{i=1}^{n} \mathbf{F}_i$, and $\mathbf{r} = \mathbf{z}_0 - \mathbf{X}_c$. This also yields

$$\tau_{z_0} = M\mathbf{r} \times \ddot{\mathbf{z}}_0 + \dot{\mathbf{L}}_{z_0},$$

where $\mathbf{L}_{z_0}$ represents the angular moment around $\mathbf{z}_0$. $M\mathbf{r} \times \ddot{\mathbf{z}}_0$ is zero if $\ddot{\mathbf{z}}_0 = \mathbf{0}$, or if $\mathbf{r} = \mathbf{0}$, or if $\mathbf{r} \parallel \ddot{\mathbf{z}}_0$.

3.3 Kinetic Energy

The kinetic energy is given by

$$T = \frac{1}{2} \sum_i m_i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i = \frac{1}{2} M \dot{\mathbf{X}}_c \cdot \dot{\mathbf{X}}_c + \frac{1}{2} \sum_i m_i \dot{\mathbf{y}}_i \cdot \dot{\mathbf{y}}_i.$$

Note that $0.5 \sum_i m_i \dot{\mathbf{y}}_i \cdot \dot{\mathbf{y}}_i = 0.5 \sum_i m_i r_i^2 \omega^2 = 0.5 I \omega^2$, and therefore

$$T = \frac{1}{2} M \left\| \dot{\mathbf{X}}_c \right\|^2 + \frac{1}{2} I \omega^2.$$

The rotational part is given by $T_{\text{rot}} = 0.5 \omega \mathbf{L}_0 = 0.5 \omega \times \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i$.

3.4 General Representation

The general representation of a system of particle is given by

$$\mathbf{M}\ddot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t) + \mathbf{F}_c,$$

where $\mathbf{F}$ represents the external forces which create work, and $\mathbf{F}_c$ represents the constraint forces. This gives us $3n$ equations but we have equally many new unknowns as we do not know the constraint forces.

Let us assume that we are given some (even non-holonomic) m constraints

$$\varphi_i(\mathbf{x}, \dot{\mathbf{x}}, t) = 0,$$

then we can differentiate these and obtain

$$\mathbf{A}(\mathbf{x}, \dot{\mathbf{x}}, t)\ddot{\mathbf{x}} = \mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t),$$

where $\mathbf{A}$ is a $3n$ by m matrix.

3.5 General Solution

In general, we the unconstrained acceleration $\mathbf{Ma} = \mathbf{F}$, and the constraint forces $\mathbf{F}_c = \mathbf{M}^{+1/2}\left(\mathbf{AM}^{-1/2}\right)^{+}(\mathbf{b} - \mathbf{Aa})$. combined, this yields the fundamental equation of motion

$$\ddot{\mathbf{x}}(t) = \mathbf{M}^{-1}\mathbf{F} + \mathbf{M}^{+1/2}\left(\mathbf{AM}^{-1/2}\right)^{+}\left(\mathbf{b} - \mathbf{AM}^{-1}\mathbf{F}\right).$$

We can find $\mathbf{M}^{-1/2}$ by fulfilling the following three conditions (i) $\mathbf{M}^{-1/2}\mathbf{M}^{1/2} = \mathbf{I}$, (ii) $\mathbf{M}^{-1/2}\mathbf{M}^{-1/2} = \mathbf{M}^{-1}$, and (iii) $\mathbf{M}^{-1/2}\mathbf{M}\mathbf{M}^{-1/2} = \mathbf{I}$. For $\mathbf{M} = \text{diag}(m_1, \ldots, m_n)$, we also have $\mathbf{M}^{-1/2} = \text{diag}(1/\sqrt{m_1}, \ldots, 1/\sqrt{m_1})$.

For a system with $\mathbf{M} = m\mathbf{I}$, we can simplify the fundamental equation to

$$\ddot{\mathbf{x}}(t) = \frac{1}{m}\mathbf{F} + \frac{1}{m}\mathbf{A}^{-1}\left(\mathbf{b} - \frac{1}{m}\mathbf{AF}\right).$$

Note that $\mathbf{e} = (\mathbf{b} - \mathbf{Aa})$ is like an error in a control equation.

3.6 Interpretation of the General Solution

We have two interpretations for this result.

Least-Squares Solution. The general solution can be interpreted as the solution to the minimization problem

$$\min_{\ddot{\mathbf{x}}} (\ddot{\mathbf{x}} - \mathbf{a})^T \mathbf{M} (\ddot{\mathbf{x}} - \mathbf{a}),$$

under the constraint $\mathbf{A}\ddot{\mathbf{x}} = \mathbf{b}$.

Nature as a controller. We realize that nature is basically a controller

$$\mathbf{\Delta}\ddot{\mathbf{x}} = \ddot{\mathbf{x}} - \mathbf{a} = \mathbf{M}^{-1/2}\left(\mathbf{A}\mathbf{M}^{-1/2}\right)^{+}(\mathbf{b} - \mathbf{A}\mathbf{a}) = \mathbf{K}_1\mathbf{e},$$

with $\mathbf{K}_1 = \mathbf{M}^{-1/2}\left(\mathbf{A}\mathbf{M}^{-1/2}\right)^{+}$. The motor command is

$$\mathbf{F}_c = \mathbf{M}^{+1/2}\left(\mathbf{A}\mathbf{M}^{-1/2}\right)^{+}(\mathbf{b} - \mathbf{A}\mathbf{a}) = \mathbf{K}_2\mathbf{e},$$

with $\mathbf{K}_2 = \mathbf{M}^{+1/2}\left(\mathbf{A}\mathbf{M}^{-1/2}\right)^{+}$.

3.7 Langrange's View of Life

1. Determine the unconstrained motion from $\mathbf{M}\ddot{\mathbf{x}}_u = \mathbf{F}$, where $\mathbf{M}$ is a n by n matrix and $\ddot{\mathbf{x}}$, $\mathbf{F}$ are n vectors.

2. Determine the constraints in the form $\mathbf{A}\ddot{\mathbf{x}} = \mathbf{b}$ where $\mathbf{A}$ is a n by m Matrix with rank $\mathbf{A} = r$.

3. Several accelerations $\ddot{\mathbf{x}}_0, \ldots, \ddot{\mathbf{x}}_{n-r}$ are possible accelerations as they fulfill $\mathbf{A}\ddot{\mathbf{x}}_i = \mathbf{b}$ where exactly $n - r$ of the $n - r + 1$ are linear independent.

4. When subtracting the $\ddot{\mathbf{x}}_0$ we obtain $n - r$ virtual displacements

   $$\delta\mathbf{x}_i = \ddot{\mathbf{x}}_i - \ddot{\mathbf{x}}_0,$$

   for which $\mathbf{A}\delta\mathbf{x}_i = \mathbf{0}$. These displacements do not create work, i.e., we have $\mathbf{F}_c^T\delta\mathbf{x}_i = 0$ (D'Alemberts principle). These can also we obtained by rewriting $\mathbf{A}\delta\mathbf{x}_i = \mathbf{0}$ as creating vectors like

   $$\mathbf{v} = \begin{bmatrix} v_1(v_{r+1}, \ldots, v_n) \\ \vdots \\ v_r(v_{r+1}, \ldots, v_n) \\ v_{r+1} \\ \vdots \\ v_n \end{bmatrix},$$

and then obtaining $1 \leq i \leq n - r$ vectors as

$$\delta \mathbf{x}_i = \begin{bmatrix} v_1(0, \dots, 0, v_{r+i}, 0, \dots, 0) \\ \vdots \\ v_r(0, \dots, 0, v_{r+i}, 0, \dots, 0) \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

5. This gives us

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{x}} &= \mathbf{F} + \mathbf{F}_c, \\ \mathbf{A}\ddot{\mathbf{x}} &= \mathbf{b}, \\ \mathbf{F}_c^T \delta \mathbf{x}_i &= 0, \end{aligned}$$

which allows us to determine the n unknowns in $\ddot{\mathbf{x}}$, and the n in $\mathbf{F}_c$.

3.8 Gauss view of Life (Le Chateliers Principle)

1. Determine the unconstrained motion from $\mathbf{M}\ddot{\mathbf{x}}_u = \mathbf{F}$, where $\mathbf{M}$ is a n by n matrix and $\ddot{\mathbf{x}}$, $\mathbf{F}$ are n vectors.

2. Nature considers all possible accelerations $\mathbf{A}\ddot{\tilde{\mathbf{x}}} = \mathbf{b}$ which comply with the constraints.

3. Nature picks the one possible acceleration which minimizes the Gaussian

$$G(\ddot{\tilde{\mathbf{x}}}) = \left(\ddot{\tilde{\mathbf{x}}} - \ddot{\mathbf{x}}_u\right)^T \mathbf{M} \left(\ddot{\tilde{\mathbf{x}}} - \mathbf{a}\right),$$

i.e., Nature is solving a global minimization problem at each instant of time or Nature takes the minimum deviation from the unconstrained motion.