Notes on Nonlinear Control

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April 27, 2004

1 Second-order Systems

1.1 Basics

Every second order system can be brought in the state-space form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, t) \\ f_2(x_1, x_2, t) \end{bmatrix}.$$

For second-order systems, we are mostly interested in the phase plane (x_2, x_1). The angle of the trajectory is given by

$$\theta(f(\mathbf{x})) = \tan^{-1} \frac{\dot{x}_2}{\dot{x}_1} = \tan^{-1} \frac{f_2(x_1, x_2, t)}{f_1(x_1, x_2, t)},$$

at a point $\vec{x}$.

1.2 Linear Systems

Every linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ can be transformed into an equivalent, canonical linear system by $\mathbf{z} = \mathbf{Q}^{-1}\mathbf{x}$, where the qualitative behavior of both is equivalent, and the one of $\mathbf{z}$ is denoted by

$$\dot{\mathbf{z}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\mathbf{z} = \mathbf{\Lambda}\mathbf{z}.$$

The matrix $\mathbf{\Lambda}$ is determined by the Eigenvalues. The Eigenvalues can be computed by

$$\lambda_{1/2} = \frac{T}{2} \pm \frac{T}{2}\sqrt{T^2 - 4\Delta},$$

where $T = \mathrm{Tr}\,\mathbf{A}$, and $\Delta = \det \mathbf{A}$. This yields several cases as shown in Table 1. The local coordinate system is then given by the

$$\mathbf{x} = \mathbf{q}_1 z_1 + \mathbf{q}_2 z_2,$$

where $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2]$. Hence you can just draw each coordinate system by by setting one of the $z_i = 0$.

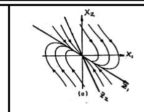
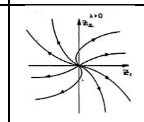
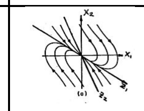
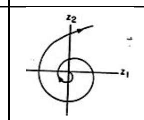
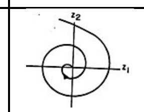
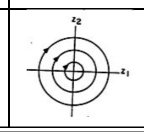
Eigenvalues	Type	Picture
$\lambda_i > 0$, and $\lambda_i \in \mathbb{R}$	Stable node	
$\lambda_i < 0$, and $\lambda_i \in \mathbb{R}$	Unstable node	
$\lambda_1 \cdot \lambda_2 < 0$, and $\lambda_i \in \mathbb{R}$	Saddle point	
$\lambda_i = \alpha \pm j\beta$, and $\lambda_i \in \mathbb{C}$	Unstable focus	
$\lambda_i = \alpha \pm j\beta$, and $\lambda_i \in \mathbb{C}$	Stable focus	
$\lambda_i = \pm j\beta_i$, and $\lambda_i \in \mathbb{C}$	Center	

Table 1: A table showing all main behaviors of second order systems

1.3 Second-order linear differential equation

The general second order linear differential equation can be given as

$$\ddot{y} + 2\xi\omega\dot{y} + \omega^2 = 0,$$

and it has the Eigenvalues $\lambda_{1/2} = -\xi\omega \pm \omega\sqrt{\xi - 2}$. This implies that it can be a stable focus ($0 < \xi < 1$), a stable node ($\xi > 1$), a center ($\xi = 0$), an unstable focus ($-1 < \xi < 0$), or an unstable node ($\xi < -1$).

1.4 Analysis by Linearization

We can linearize the system around an equilibrium point $\mathbf{x}_{\text{eq}}$ for which $f(\mathbf{x}_{\text{eq}}) = 0$. When defining $\boldsymbol{\Delta}\mathbf{x} = \mathbf{x} - \mathbf{x}_{\text{eq}}$, we can linearize the local solution and get

$$\Delta\dot{x}_1 = \frac{\partial f_1}{\partial x_1}\Delta x_1 + \frac{\partial f_1}{\partial x_2}\Delta x_2 + \epsilon_1,$$
$$\Delta\dot{x}_2 = \frac{\partial f_2}{\partial x_1}\Delta x_1 + \frac{\partial f_2}{\partial x_2}\Delta x_2 + \epsilon_2,$$

where $\epsilon \in O(\Delta x^2)$. For small Δx_i the linearized system yields similar behavior as the nonlinear one except for the center.

1.5 Drawing Phase Planes

This gives a recipe for drawing phase planes:

1. Horizontal axis is denoted by x_1, vertical axis by x_2.
2. Equilibrium Points
   (a) Determine equilibrium points by $f(\mathbf{x}_{\mathrm{eq}}) = 0$.
   (b) Linearize around equilibrium points by $A_{ij} = \partial f_i / \partial x_j$.
   (c) Determine equilibrium point type from Eigenvalues by
   $$\lambda_{1/2} = \frac{T}{2} \pm \frac{T}{2}\sqrt{T^2 - 4\Delta},$$
   where $T = \operatorname{Tr}\mathbf{A}$, and $\Delta = \det\mathbf{A}$
   (d) Determine local coordinate system or Eigenvectors from Table 1.
   (e) Draw local trajectories using the example.
3. Determine Isoclines
   (a) Vertical arrows up where $\dot{x}_1 = 0$, $\dot{x}_2 > 0$.
   (b) Vertical arrows down where $\dot{x}_1 = 0$, $\dot{x}_2 < 0$.
   (c) Horizontal arrows to the right where $\dot{x}_2 = 0$, $\dot{x}_1 > 0$.
   (d) Horizontal arrows to the left where $\dot{x}_2 = 0$, $\dot{x}_1 < 0$.
4. Use symmetry if possible.
5. Draw Trajectories which do not intersect.

1.6 Special case: $\ddot{y} = g(\dot{y}, y)$

The system $\ddot{y} = g(\dot{y}, y)$ becomes $\dot{x}_1 = \dot{y} = x_2$, $\dot{x}_2 = g(\dot{y}, y) = g(x_2, x_1)$ in state-space form. It has the following properties:

- All equilibrium points intersect with the horizontal axis.
- All trajectories have vertical slope at the horizontal axis.
- If $|g(\dot{y}, y)|$ is bounded, the vertical slope can only occur on the horizontal axis.

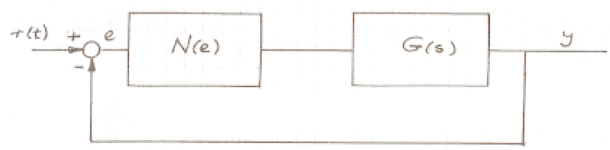


Figure 1: Separation of the linear and nonlinear block.

2 Piecewiese linear Switching Systems

2.1 Response to a Step Impulse

The response to a step input can be analyzed as follows:

1. Seperate the system into a nonlinear system $N(e, \dot{e})$ and a linear system $F(s)$ as in Figure 1.

2. Transform the $F(s) = Y(s)/U(s)$ into time-domain where $f(y, \dot{y}, \ddot{y}) + u = 0$.

3. The step input $r(t) = A \cdot 1(t)$, yields $e = r - y = A - y$,which implies

$$y = A - e,\ \dot{y} = -\dot{e},\ \ddot{y} = -\ddot{e},$$

   and therefore

$$f(A - e, -\dot{e}, -\ddot{e}) + N(e) = 0.$$

4. Separate all linear regions of $N(e)$ by decision borders, and rename the $x_1 = e$, and $x_2 = \dot{e}$.

5. Analyze the different piece-wise linear regions. Use separation of variable for

$$\frac{dx_2}{dx_1} = h(x_1, x_2),$$

   if possible. Three practical cases:(i) Lines $h(x_1, x_2) = const$, (ii) Parabola $h(x_1, x_2) = B/x_2$.

2.2 Response to a Ramp Impulse

The response to a step input can be analyzed as follows:

1. Seperate the system into a nonlinear system $N(e, \dot{e})$ and a linear system $F(s)$ as in Figure 1.

2. Transform the $F(s) = Y(s)/U(s)$ into time-domain where $f(y, \dot{y}, \ddot{y}) + u = 0$.

3. The step input $r(t) = At$, yields $e = At - y$,which implies

$$y = At - e,\ \dot{y} = A - \dot{e},\ \ddot{y} = -\ddot{e},$$

and therefore

$$f(At - e, A - \dot{e}, -\ddot{e}) + N(e) = 0.$$

4. Separate all linear regions of $N(e)$ by decision borders, and rename the $x_1 = e$, and $x_2 = \dot{e}$.

5. Analyze the different piece-wise linear regions. Use separation of variable for

$$\frac{dx_2}{dx_1} = h(x_1, x_2),$$

if possible. Three practical cases:(i) Lines $h(x_1, x_2) = const$, (ii) Parabola $h(x_1, x_2) = B/x_2$.

3 Conservative Systems

3.1 Basics

Aconservative system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -f(x_1) \end{bmatrix},$$

which implies

$$\frac{dx_2}{dx_1} = -\frac{f(x_1)}{x_2} \Rightarrow \frac{x_2^2}{2} + \int_{x_{10}}^{x_1} f(x_1)dx_1 = T + V = E,$$

where T is kinetic energy, V is potential energy, and E the total energy determined by the initial conditions. Trajectories can be determined by $x_2 = \pm\sqrt{2\left(E - V(x_1)\right)}$.

3.2 Interesting Notes

Interestingly, we have:

- The equilibrium points are at the extrema of the potential energy

$$f(x_1) = \frac{\partial V}{\partial x_1} = 0.$$

- Maxima of $V(x_1)$ are stable equilibrium points.
- Minima of $V(x_1)$ are instable equilibrium (saddle) points.

4 Describing Function Method

4.1 Basics

DFM is employed to determine self-oscillation.

1. Assume a time-invariant $N(e)$, $e = \hat{e} \sin \omega t$.

2. Determine $y(t)$ by Fourier series expansion while neglecting bias (α_0 or $k = 0$), and higher order harmonics ($k > 2$). The Fourier coeffiecients are

$$\alpha_1 = \int_{t_0}^{t_0+T} y(t) \cos(\omega t) dt,$$
$$\beta_1 = \int_{t_0}^{t_0+T} y(t) \sin(\omega t) dt.$$

3. This yields

$$y(t) = \alpha_1 \cos \omega t + \beta_1 \sin \omega t = (\beta_1 + \alpha_1 j) \sin \omega t = N_1 \sin \omega t,$$

   where $N_1 = \sqrt{\alpha_1^2 + \beta_1^2} e^{j\phi}$ with $\phi = \tan(\alpha_1/\beta_1)$.

4. Determine the equivalent gain or describing function

$$\eta(\hat{e}, \omega) = \frac{1}{\hat{e}} \sqrt{\alpha_1^2 + \beta_1^2} e^{j\phi} = \frac{1}{\hat{e}} (\beta_1 + \alpha_1 j).$$

   This yields the output oscillations

$$y(t) = \eta(\hat{e}, \omega) \hat{e} \sin \omega t.$$

4.2 Properties

We have proved the following properties in the lecture:

- A sinusoidal describing function for a memoryless nonlinearity is always real,i.e., $\alpha_1 = 0$.

- If nonlinear characteristic $N(e)$ is memoryless, and time-invariant, then the chracteristic function is independant of the frequency $\eta(\hat{e}, \omega) = \eta(\hat{e})$.

4.3 Analysis of Limit Cycles with DFM

The input is assumed to be a pure sinusoid, and higher order harmonic effects are neglected. We call the linear system $G(s)$. This allows the following recipe:

1. Replace $N(e)$ by $\eta(\hat{e}, \omega)$. Determine $G(j\omega) = G(s)|_{j\omega}$.

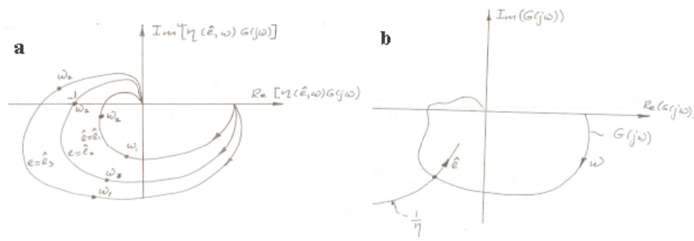


Figure 2: The Family of Nyquest Plots methods is described in (a), and the Nyquest plane of $G(j\omega)$ is shown in (b).

2. A limit cycle exists if
$$G(j\omega) = -\frac{1}{\eta(\hat{e}, \omega)},$$
or equivalently
$$\eta_R G_R + \eta_I G_I = -1,$$
$$\eta_R G_I + \eta_I G_R = 0.$$

3. This can be solved either by any of the following methods.

Family of Nyquest Plots: Plot imaginary component $\mathrm{Im}[G(j\omega)\eta(\hat{e}, \omega)]$ versus real component $\mathrm{Re}[G(j\omega)\eta(\hat{e}, \omega)]$ for different $\hat{e}$, and find the $\hat{e}_{-1}$ for which $G(j\omega)\eta(\hat{e}_{-1}, \omega) = -1$. The ω for which this is true is the frequency of the cycle. See Figure 2 (a).

Nyquest plane of $G(j\omega)$: Plot $G(j\omega)$ as functions of ω, and $-1/\eta(\hat{e})$ as functions of $\hat{e}$. The ω, and $\hat{e}$ where they meet are frequency and amplitude of the cycle, respectively. See Figure 2 (b).

Analytical method: For memory-free nonlinearities, we can solve the equation $\eta_R(\hat{e})G_I(j\omega) = 0$ for $\omega = \omega_c$. We substitute this and solve $\eta_R(\hat{e})G_R(j\omega_c) = -1$ for $\hat{e} = \hat{e}_c$.

4. Determine closed loop stability as shown in Figure 3, or using
$$\max_{\hat{e}} \left(-\frac{1}{\eta(\hat{e})} \right) > G(j\omega_C).$$

5 Lyopanov Equilibrium Point Analysis

5.1 Definitions

Equilibrium Point: The point $\mathbf{x}_0$ is an EQ iff $f(\mathbf{x}_0, t) = 0$.

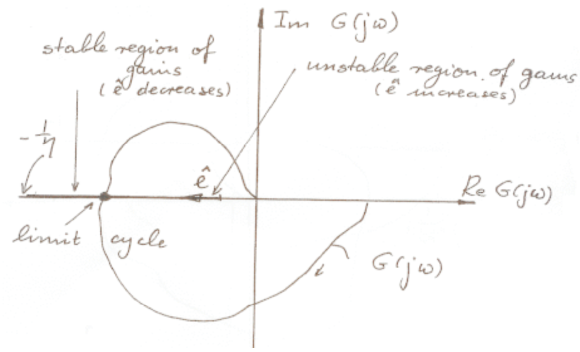


Figure 3: This figure shows the stability analysis for a system with a memory-free nonlinearity.

Isolated EQP: $\mathbf{x}_0$ is an isolated EQ iff $\mathbf{x}_0$ EQ$\wedge\neg\exists\mathbf{x} \in B_\epsilon(\mathbf{x}_0) : f(\mathbf{x}, t) = 0$.

Lyopanov function: Scalar function $V(\mathbf{x}, t)$, derivbative is $\dot{V}(\mathbf{x}, t) = \partial V/\partial t + (\partial V/\partial \mathbf{x})^T f(\mathbf{x}, t)$.

LDPF: $\forall \mathbf{x} \in B_\epsilon(\mathbf{x}_0) : V(\mathbf{x}, t) \geq W(\mathbf{x}) > 0$, and $V(\mathbf{0}, t) = W(\mathbf{0}) = 0$.

PDF: $\forall \mathbf{x} \in \mathbb{R}^n : V(\mathbf{x}, t) \geq W(\mathbf{x}) > 0$, and $V(\mathbf{0}, t) = W(\mathbf{0}) = 0$.

Radially unbounded: $\lim_{\|\mathbf{x}\| \to \infty} W(\mathbf{x}) \to \infty$.

Decrescent: $\forall \mathbf{x} \in \mathbb{R}^n, t \geq 0 : V(\mathbf{x}, t) \leq \hat{W}(\mathbf{x})$.

5.2 Stability

Stability: An EQ is stable if

$$\forall \epsilon > 0 : \exists \delta(t_0, \epsilon) > 0 : \|\mathbf{x}(t_0)\| < \delta(t_0, \epsilon) \implies \|\mathbf{x}(t)\| < \epsilon . \forall t \geq t_0.$$

This is implied by: V lpdf, and $\dot{V}(\mathbf{x}, t) \leq 0$.

Uniform Stability: An EQ is uniformly stable if

$$\forall \epsilon > 0 : \exists \delta(\epsilon) > 0 : \|\mathbf{x}(t_0)\| < \delta(\epsilon) \implies \|\mathbf{x}(t)\| < \epsilon . \forall t \geq t_0.$$

This is implied by: V decrescrent, lpdf, and $\dot{V}(\mathbf{x}, t) \leq 0$.

Asymptotic stability: An EQ is asymptotically stable iff $\exists \delta > 0 : \|\mathbf{x}(t_0)\| < \delta \implies \lim_{t \to \infty} \|\mathbf{x}(t)\| = 0$. This is implied by: V decrescrent, lpdf, and $-\dot{V}(\mathbf{x}, t)$ lpdf.

Global Asymptotic Stability: V decrescrent, pdf, radially unbounded, and $-\dot{V}(\mathbf{x}, t) \leq -\hat{W}(\mathbf{x})$.

Exponential Stability: A EQ is exponentially stable iff $\exists r, b, a > 0 : \|\mathbf{x}(t)\| \leq k\|\mathbf{x}(t_0)\| e^{-bt}. \forall t \geq t_0$. This is given by: $\forall \mathbf{x} \in B_\epsilon(\mathbf{x}_0) : a\|\mathbf{x}\|^p \leq V(\mathbf{x}, t) \leq b\|\mathbf{x}\|^p$, $\dot{V}(\mathbf{x}, t) \leq -c\|\mathbf{x}\|^p$.

Global Exponential Stability: $\forall \mathbf{x} \in \mathbb{R}^n : a\|\mathbf{x}\|^p \leq V(\mathbf{x}, t) \leq b\|\mathbf{x}\|^p$, $\dot{V}(\mathbf{x}, t) \leq -c\|\mathbf{x}\|^p$.

5.3 Further Lyopanov Methods

Instability theorem: Choose a V so that $+\dot{V}$ is lpdf, and $V(\mathbf{0}, t) = 0$. Show that $V(\mathbf{x}, t) > 0$ for any point $\mathbf{x}$ which is arbitrarily close to the origin.

La-Salle Krakovski: $\mathbf{x} = \mathbf{0}$ is asymptotically stable if (i) $V(\mathbf{x})$ lpdf, (ii) $\Omega_e = \{\mathbf{x} | V(\mathbf{x}) \leq c\}$ is bounded, (ii) $\dot{V}(\mathbf{x}) \leq 0$, and (iv) the set $S = \{\mathbf{x} \in \Omega_e | \dot{V}(\mathbf{x}) = 0\}$ contains no trajectories of the system (e.g., $V(0, x_2) = 0$, and $x_1 = 0 \Longrightarrow \dot{x}_2 \neq 0$).

Linear Time-Invariant Systems: For a time-invariant system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, the origin is globally asymptotically stable iff equivalently (i) $\forall i. \operatorname{Re} \lambda_i(\mathbf{A}) < 0$, or (ii) given a positive definite symmetric matrix Q, we can solve

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} = -\mathbf{Q}$$

for a unique, positive definite $\mathbf{P}$. (Note: Solving for $\mathbf{P}$ given $\mathbf{Q}$ is sufficient and necessary, solving for $\mathbf{Q}$ given $\mathbf{P}$ is *only* sufficient and *not* necessary).

Linear Time-Variant Systems: For the time-variant system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, defining $\mathbf{H}(t) = \mathbf{A}(t) + \mathbf{A}^T(t)$. Then (sufficient but not necessary):

1. State is bounded by:

$$\|\mathbf{x}(0)\| \frac{1}{2} \int_{t_0}^{t} \lambda_{\min}(\mathbf{H}(\tau)) d\tau \leq \|\mathbf{x}(t)\| \leq \|\mathbf{x}(0)\| \frac{1}{2} \int_{t_0}^{t} \lambda_{\max}(\mathbf{H}(\tau)) d\tau.$$

2. Stability: $\lim_{t\to\infty} \int_{t_0}^{t} \lambda_{\min}(\mathbf{H}(\tau)) d\tau < M(t_0) < \infty$, unifrom stability for $M(t_0) = M$.
3. Uniform asymptotic stability: $\lim_{t\to\infty} \int_{t_0}^{t} \lambda_{\min}(\mathbf{H}(\tau)) d\tau = -\infty$.
4. Instable: $\lim_{t\to\infty} \int_{t_0}^{t} \lambda_{\min}(\mathbf{H}(\tau)) d\tau = +\infty$.

Lyopanovs Indirect Method: The system is given as $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + f_1(\mathbf{x}, t)$, where $\mathbf{A}(t) = \partial f / \mathbf{x}|_{\mathbf{x}=\mathbf{0}}$, $f_1(\mathbf{x}, t) = \mathbf{A}(t) - f(\mathbf{x}, t)$, and

$$\lim_{\|\mathbf{x}\|\to 0} \sup_{t>0} \frac{f_1(\mathbf{x}, t)}{\|\mathbf{x}\|} = 0.$$

This can be solved by performing a stability analysis for the linear system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$. Asymptotic (Exponential/In-) stability of the linearized system implies local asymptotic (exponential/In-) stability of the nonlinear system. We can make use of this for finding stable feedback controllers.

Determine Region of Attraction: Determine (i) a compact set Ω_e containing $\mathbf{x}_e$ such that it is invariant under $\dot{\mathbf{x}} = f(\mathbf{x})$ (for example $\Omega_e = \{\mathbf{x} | V(\mathbf{x}) \leq c \vee \dot{V}(\mathbf{x}) < 0\}$), and (ii) $\forall \mathbf{x} \neq 0 : \dot{V}(\mathbf{x}) < 0$ and $\dot{V}(\mathbf{0}) = 0$. Follows from LaSalle-Krasovski.

5.4 Helpful Lyopanov Proofs

Possible Lyopanov functions: (i) use the total energy of the physical system, (ii) use $V(\mathbf{x}) = \sum_i x_i^2/2 \Longrightarrow \dot{V}(\mathbf{x}) = \sum_i x_i \dot{x}_i$,

Mechanical Systems: Given a mechanical system $m\ddot{x} + f(\dot{x}) + g(x) = 0$, with continous f, g, so that $\forall x. x f(x) \geq 0$, $\forall x. x g(x) \geq 0$, we can use $V(x) = x_2^2/2 + \int_0^{x_1} g(\sigma) d\sigma \Longrightarrow \dot{V}(x) = -x_2 f(x_2) \leq 0$. Stability follows through LaSalle-Krasovski.

Exponential stability: Use $V(\mathbf{x}) = \mathbf{x}^T \mathbf{Q} \mathbf{x}$, then we have $\lambda_{\min}(\mathbf{Q}) \|\mathbf{x}\|^2 \leq V(\mathbf{x}, t) \leq \lambda_{\max}(\mathbf{Q}) \|\mathbf{x}\|^2$, you only have to determine $\dot{V}(\mathbf{x}, t) \leq -c \|\mathbf{x}\|^p$.

Instability: Sometimes, $V(\mathbf{x}) = x_1^2 - x_2^2$ is practical iff $-\dot{V}(\mathbf{x}) = -x_1 \dot{x}_1 + x_2 \dot{x}_2 > 0$.

5.5 Notes

- Stability is defined in terms if equilibrium points and NOT systems.
- Lyopanov stability notions are local.
- An EQ can only be stable or unstable.
- For an autonomous system, a stable EQ is also uniformly stable.
- Lyopanov functions are not unique.
- Interpretation: going downhill in direction $\dot{\mathbf{x}}$ in a landscape $V(\mathbf{x})$ within $90°$ of the steepest descent $-\partial V(\mathbf{x})/\partial \mathbf{x}$, i.e., at $-\dot{V}(\mathbf{x}) = -(\partial V(\mathbf{x})/\partial \mathbf{x})^T \dot{\mathbf{x}} \geq 0$.

5.6 Applications of Lyaponov Theory

Feedback stabilization: For a system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$, $\mathbf{y} = g(\mathbf{x})$, we intend to find $\mathbf{u} = -p(\mathbf{y}, t)$ so that $\lim_{t \to \infty} \mathbf{e}(t) = 0$ where $\mathbf{e}(t) = \mathbf{y}(t) - \mathbf{y}_d$. We can then simply stabilize the system $\dot{\mathbf{x}} = f(\mathbf{x}, -p(g(\mathbf{x}), t), t)$. *Example*: For mechanical systems $\dot{\mathbf{x}} = f(\mathbf{x}, t) + M(\mathbf{x})\mathbf{u}$, we can choose $\mathbf{u} = -M^{-1}(\mathbf{x})[f(\mathbf{x}, t) - h(\mathbf{x})]$, which turns the system into $\dot{\mathbf{x}} = h(\mathbf{x})$ where h is chosen stable.

Trajectory following: When following a trajectory $\dot{\mathbf{x}}_d = h(\mathbf{x}_d)$ with a system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$, and a controller $\mathbf{u} = p_1(\mathbf{x}_d - \mathbf{x}) + p_2(\mathbf{x}_d)$. We can then choose $\xi = [\mathbf{x}_d, \mathbf{x} - \mathbf{x}_d] = [\mathbf{x}_d, \mathbf{e}]$ which yields $\dot{\xi} = [h(\mathbf{x}_d), f(\mathbf{x}, \mathbf{u}, t)] =$

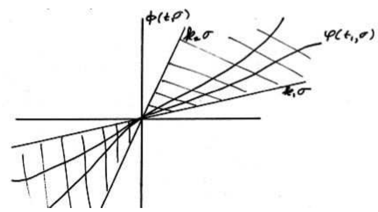


Figure 4: This figure shows the illustration for a sector.

$[h(\mathbf{x}_d), f(\mathbf{e}+\mathbf{x}_d, p_1(\mathbf{e})+p_2(\mathbf{x}_d), t)]$. We then choose the controller so that the origin is asymptotically stable.

Adaptive control: We have a system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{r}$, and a (desired) model $\dot{\mathbf{x}}_m = \mathbf{A}_m\mathbf{x}_m + \mathbf{B}_m\mathbf{r}_m$. By defining $\mathbf{e} = \mathbf{x}_m - \mathbf{x}$, we obtain $\dot{\mathbf{e}} = \mathbf{A}_m\mathbf{e} + (\mathbf{A}_m - \mathbf{A})\mathbf{x} + (\mathbf{B}_m - \mathbf{B})\mathbf{r}$. Using the Lyopanov function $V(\mathbf{e}) = \mathbf{e}^T\mathbf{P}\mathbf{e} + \sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij}\alpha_{ij}^2 + \sum_{i=1}^{n}\sum_{j=1}^{m}\nu_{ij}\beta_{ij}^2$ with $\mu_{ij}, \nu_{ij} > 0$, $[\alpha_{ij}] = (\mathbf{A}_m - \mathbf{A})$, $[\beta_{ij}] = (\mathbf{B}_m - \mathbf{B})$, and using the update rules

$$\dot{\alpha}_{ij} = -\frac{1}{\mu_{ij}}x_j\mathbf{e}^T\mathbf{p}_i,\ \dot{\beta}_{ij} = -\frac{1}{\nu_{ij}}x_j\mathbf{e}^T\mathbf{p}_i$$

in $\dot{V}$ (where $\mathbf{P} = [\mathbf{p}_1, \ldots, \mathbf{p}_2]$), we obtain $\dot{V} = -\mathbf{e}^T\mathbf{e}$. The error of the trajectory goes to zero but not the model parameters.

6 Frequency Domain Methods

The Lure Problem: The system is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t),\ y(t) = \mathbf{c}^T\mathbf{x}(t) + du(t),\ u(t) = -\varphi(y(t)).$$

The linearity is equivalent to $G(s) = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d$.

Assumptions: We assume that either $\mathbf{A}$ is Hurwitz ($\forall i. \operatorname{Re}\lambda_i(\mathbf{A}) < 0$ or exactly one Eigenvalue is zero $\lambda_1(\mathbf{A}) = 0$, $\forall i \geq 2. \operatorname{Re}\lambda_i(\mathbf{A}) < 0$). $(\mathbf{A}, \mathbf{b})$ controllable, $(\mathbf{A}, \mathbf{c})$ observable. The nonlinearity is either odd ($\varphi(0) = 0$, $\forall y \neq 0. y\varphi(y) > 0$), or in a sector (see below).

φ in a sector: φ belongs to a sector $[k_1, k_2]$, ($\varphi \in S(k_1, k_2)$), if

$$k_1y^2 \leq y\varphi(y, t) \leq k_2y^2.$$

Stability in a sector is called the absolute stability problem. See Figure 4 for an illustration.

Aizerman's conjecture: If $d = 0$, $\varphi \in S(k_1, k_2)$, $\forall k \in [k_1, k_2].(\mathbf{A} - \mathbf{b}k\mathbf{c}^T)$ is Hurwitz, then the origin is globally asymptotically stable.

Kalman's conjecture: If $\forall k \in [k_1, k_2].(\mathbf{A} - \mathbf{b}k\mathbf{c}^T)$ is Hurwitz, and if

$$k_1 \leq \frac{\partial \varphi(y,t)}{\partial y} \leq k_2,$$

then the origin is globally asymptotically stable (stricter than Aizerman's conjecture).

Kalman-Yakubovitch lemma: If $\mathbf{A}$ is Hurwitz, $(\mathbf{A}, \mathbf{b})$ controllable, $\mathbf{v} \in \mathbb{R}^n$, $\gamma \geq 0$, $\varepsilon > 0$, $\mathbf{Q}$ pd, then there is pd $\mathbf{P}$, and $\mathbf{q} \in \mathbb{R}^n$, so that

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{q}\mathbf{q}^T - \varepsilon\mathbf{Q},\ \mathbf{P}\mathbf{b} - \mathbf{v} = \sqrt{\gamma}\mathbf{q},$$

if and only if ε small, and $h(s) = \gamma + 2\mathbf{v}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ satisfies $\forall \omega \in \mathbb{R}. \operatorname{Re} h(j\omega) > 0$.

Circle criterion: If $\varphi \in S(\alpha, \beta)$, the origin of the system will be absolutely stable/globally asymptotically stable if one of the following four sufficient graphical conditions applies (these are illustrated in Table 2).

1. If $0 < \alpha < \beta$, the nyquist plot has to encircle the system m times in counterclockwise direction, where m is the number of poles with positive real part.
2. If $0 = \alpha < \beta$, the nyquist plot lies above $-1/\beta$.
3. If $\alpha < 0 < \beta$, the nyquist plot lies in the interior of the disk $D(\alpha, \beta)$.
4. If $\alpha < \beta < 0$, then use $\hat{g} = -g$, $\hat{\alpha} = -\beta$, $\hat{\beta} = -\alpha$, and apply (1).

Alternatively, one could interprete it analytically as $\phi \in S(0, k)$, $1 + kd > 0$, $\operatorname{Re}\{1 + kg(j\omega)\} > 0 \Longrightarrow$absolute stability.

Popov criterion: The system is slightly modified to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u,\ y = \mathbf{c}^T\mathbf{x} + d\xi,\ \dot{\xi} = u,\ u = -\varphi(y),$$

i.e., with a transfer function $G(s) = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d/s$, which has to be Hurwitz, or one Eigenvalue that is zero. The system is globally asymptotically stable if there exists a number r such that $\operatorname{Re}\{1 + j\omega r g(j\omega)\} + 1/k > 0$, or equivalently

$$\operatorname{Re}\{g^*(j\omega)\} > -\frac{1}{k} + r \operatorname{Im} g^*(j\omega),$$

with $g^*(j\omega) = \operatorname{Re} g(j\omega) + j\omega \operatorname{Im} g(j\omega)$. The graphical interpretation is given in Figure 5.

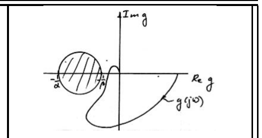
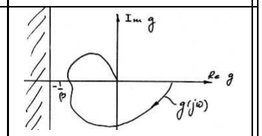
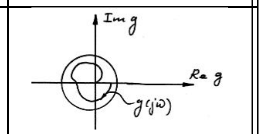
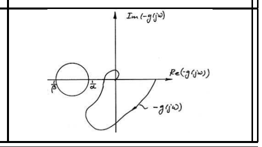
	Disk $D(\alpha, \beta)$	Hurwitz?	Sketch
Case (1)	$0 < \alpha < \beta$	-	
Case (2)	$0 = \alpha < \beta$	Required	
Case (3)	$\alpha < 0 < \beta$	Required	
Case (4)	$\alpha < \beta < 0$	-	

Table 2: This table shows the four cases of the circle criterion.

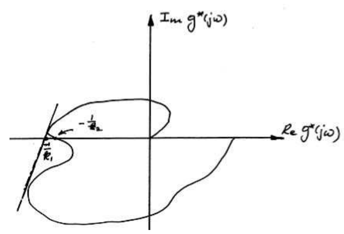


Figure 5: This figure shows the Popov criterion.

7 Extensions of the Notion of Stability

7.1 Boundedness

Uniformly bounded: $\forall a \in (0, c).\exists b(a).\forall t \geq t_0.\, \|x(t_0)\| \leq a \Longrightarrow \|x(t)\| < b$.

Globally uniformly bounded: $\forall a \in \mathbb{R}.\exists b(a).\forall t \geq t_0.\, \|x(t_0)\| \leq a \Longrightarrow \|x(t)\| < b$.

Uniformly ultimately bounded: $\forall a \in \mathbb{R}.\exists b(a).\forall t \geq t_0 + T(a, b).\, \|x(t_0)\| \leq a \Longrightarrow \|x(t)\| < b$.

To be continued...

7.2 Perturbations

Additive perturbation: We assume that the system can be split into a nominal model f and an additive perturbation g, so that

$$\dot{\mathbf{x}} = f(\mathbf{x}, t) + g(\mathbf{x}, t) = f(\mathbf{x}, t) + (f(\mathbf{x}, t) - \bar{f}(\mathbf{x}, t)).$$

Any same/lower order perturbation can be represented like this.

Vanishing perturbations: The origin is an equilibrium point of the perturbation, i.e., $g(\mathbf{x}, t) = 0$.

Robustness of exponential stability: Assume $\mathbf{x} = 0$ an exponentially stable equilibrium point of f, if f is exponentially stable on D, i.e.,

$$\forall \mathbf{x} \in D. c_1 \|\mathbf{x}\|^2 \leq V(\mathbf{x}, t) \leq c_2 \|\mathbf{x}\|^2, c_3 \|\mathbf{x}\|^2 \leq -\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}}^T f(\mathbf{x}, t)\right),$$

and $\forall \mathbf{x} \in D. \|\partial V / \partial \mathbf{x}\| < c_4 \|\mathbf{x}\|^2$. If furthermore, the perturbation fulfills $\forall \mathbf{x} \in D. \|g(\mathbf{x}, t)\| < \gamma \|\mathbf{x}\|$ with $\gamma < c_3/c_4$, the system is exponentially stable. If $D = \mathbb{R}^n$, even globally.

Linear Systems Nonlinearly Perturbed: Assume $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + g(\mathbf{x}, t)$, $\mathrm{Re}\{\lambda_i(\mathbf{A})\} < 0$, and $\|g(\mathbf{x}, t)\| < \gamma \|\mathbf{x}\|$. If $\mathbf{P}$ is a solution of the Lyopanov equation $\mathbf{PA} + \mathbf{A}^T\mathbf{P} = -\mathbf{Q}$, and $\gamma < \lambda_{\min}(\mathbf{Q})/(2\lambda_{\max}(\mathbf{P}))$, the system will be globally exponetially stable. This ratio γ is maximal for $\mathbf{Q} = \mathbf{I}$.

Robustness of Asymptotic Stability: Assume $\mathbf{x} = 0$ an asymptotically stable equilibrium point of f, i.e.,

$$W_1(\mathbf{x}) \leq V(\mathbf{x}, t) \leq W_2(\mathbf{x}), c_3 W_{3/4}^2(\mathbf{x}) \leq -\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}}^T f(\mathbf{x}, t)\right),$$

and $\|\partial V / \partial \mathbf{x}\| < c_4 W_{3/4}(\mathbf{x})$; all $W_i(\mathbf{x})$ are pdf. If furthermore, the perturbation fulfills $\|g(\mathbf{x}, t)\| < \gamma W_{3/4}(\mathbf{x})$ with $\gamma < c_3/c_4$, the system is asymptotically stable.

Nonvanishing Perturbations: Assume $\mathbf{x} = 0$ an exponentially stable equilibrium point of f, if f is exponentially stable on $D = \{\mathbf{x} \in \mathbb{R}^n | \|\mathbf{x}\| < r\}$, i.e.,

$$\forall \mathbf{x} \in D. c_1 \|\mathbf{x}\|^2 \leq V(\mathbf{x}, t) \leq c_2 \|\mathbf{x}\|^2, c_3 \|\mathbf{x}\|^2 \leq -\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}}^T f(\mathbf{x}, t)\right),$$

and $\forall \mathbf{x} \in D. \|\partial V / \partial \mathbf{x}\| < c_4 \|\mathbf{x}\|^2$. If furthermore, the perturbation fulfills

$$\forall \mathbf{x} \in D. \|g(\mathbf{x}, t)\| \leq \delta < \frac{c_3}{c_4} \sqrt{\frac{c_1}{c_2}} \theta r,$$

for some $0 < \theta < 1$. Then for all $\|\mathbf{x}(0)\| < r\sqrt{c_1/c_2}$,

$$\forall t_0 \le t \le T.\, \|\mathbf{x}(t)\| \le k \exp\left[-\gamma(t - t_0)\right] \|\mathbf{x}(0)\| \wedge \|\mathbf{x}(t)\| \le b,$$

for finite T, where $k = \sqrt{c_2/c_1}$, $\gamma = (1-\theta)c_3/(2c_2)$, $b = (c_3/c_4)\sqrt{c_1/c_2}(\delta/\theta)$.

8 Nonlinear Control Design

8.1 Introduction to Feedback Control Design

Objectives of Control Design: Equilibrium point stabilization, Trajectory tracking, Disturbation rejection (input/output boundedness), robustness (cope with model errors).

State feedback stabilization: Show that the desired equilibrium point in

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t),\ \mathbf{u} = \gamma(\mathbf{x}, \mathbf{z}, t),\ \dot{\mathbf{z}} = f(\mathbf{x}, \mathbf{z}, t),$$

is asymptotically stable. Use $\mathbf{z}$ in order to implement I-controllers.

Output feedback stabilization: Show that the desired equilibrium point in

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t),\ \mathbf{y} = h(\mathbf{x}, \mathbf{u}, t),\ \mathbf{u} = \gamma(\mathbf{y}, \mathbf{z}, t),\ \dot{\mathbf{z}} = f(\mathbf{y}, \mathbf{z}, t),$$

is asymptotically stable. Use $\mathbf{z}$ in order to implement I-controllers. For moving the equilibrium point, all variables $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$, $\hat{\mathbf{u}} = \mathbf{u} - \mathbf{u}_0$, $\hat{\mathbf{y}} = h(\mathbf{x}_0 + \hat{\mathbf{x}}, \mathbf{u}_0 + \hat{\mathbf{u}}, t) - h(\mathbf{x}_0, \mathbf{u}_0, t)$, have to be moved.

Linear Systems: For linear time-invariant systems ($\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$, $\mathbf{u} = -\mathbf{K}\mathbf{x}$), both problems become easier. State feedback linearization requires only $\operatorname{Re}\lambda_i(\mathbf{A} - \mathbf{B}\mathbf{K}) < 0$. Output stabilization requires an Observer $d\hat{\mathbf{x}}/dt = (\mathbf{A} - \mathbf{B}\mathbf{K})\hat{\mathbf{x}} + \mathbf{L}(\mathbf{x} - \mathbf{C}\hat{\mathbf{x}} - \mathbf{D}\mathbf{u})$, so that we analyze

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}.$$

where $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ represents the error.

Stabilization by Linearization: We can stabilize an equilibrium point of the nonlinear system by linearization. We can determine the region of stability using a quadratic Lyopanov function.

8.2 Closed-loop Control

To be continued...

8.3 Integral Control / Gain Scheduling

System description: The system is described by

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{w}),\ \dot{\sigma} = h(\mathbf{x}, \mathbf{w}) - \mathbf{r},$$

where $\mathbf{w}$ is unknown.

Stabilizing Control Law: A stabilizing control law $\mathbf{u} = \gamma(\mathbf{x}, \sigma, h(\mathbf{x}, \mathbf{w}) - \mathbf{r})$ has to fulfill (i) $\mathbf{u}_0 = \gamma(\mathbf{x}_0, \sigma_0, 0)$, and (ii) that $\dot{\mathbf{x}} = f(\mathbf{x}, \gamma(\mathbf{x}, \sigma, h(\mathbf{x}, \mathbf{w}) - \mathbf{r}), \mathbf{w})$, $\dot{\sigma} = h(\mathbf{x}, \mathbf{w}) - \mathbf{r}$, has to have an asymptotically stable equilibrium point at $(\mathbf{x}_0, \mathbf{u}_0)$.

Via Linearization: Use the linear control law $\mathbf{u} = -\mathbf{K}_1\mathbf{x} - \mathbf{K}_2\sigma - \mathbf{K}_3\mathbf{e}$ with error $\mathbf{e} = h(\mathbf{x}, \mathbf{w}) - \mathbf{r}$. If $\mathbf{K}_2$ is nonsingular, there is a unique solution for σ_0 so that $\mathbf{u}_0 = -\mathbf{K}_1\mathbf{x}_0 - \mathbf{K}_2\sigma_0$.

Gain scheduling: For tracking a trajectory $r(t)$, we treat each point of the trajectory as an equilibrium point, i.e., we have gains depending on r so that $\mathbf{u} = -\mathbf{K}_1(r)\mathbf{x} - \mathbf{K}_2(r)\sigma - \mathbf{K}_3(r)\mathbf{e}$ is stable if r was constant.

8.4 Sliding Mode Control

Objective: Assume a system

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = h(x_1, x_2) + g(x_1, x_2)u,$$

where g, h are unknown and $g(x_1, x_2) \geq g(x_{10}, x_{20}) > 0$. We intend bring the system to the manifold $s = a_1x_1 + x_2 = 0$ with arbitrary a_1.

Control Law: Assume that you know

$$\left\| \frac{a_1x_2 + h(x_1, x_2)}{g(x_1, x_2)} \right\| \leq \rho(x_1, x_2),$$

and the Lyopanov function $V(s) = 0.5s^2$, and $\dot{V}(s) = s\dot{s} \leq g(x_1, x_2)|s|\rho(x_1, x_2) + s \cdot g(x_1, x_2)u$. Then

$$u = -\beta(x_1, x_2)\,\mathrm{sign}\, s,$$

is a stable control law if $\beta(x_1, x_2) > \rho(x_1, x_2) + \beta_0$ for some $\beta_0 > 0$. Nota bene: (i) Manifold is reached in finite time, and $x_2 = \dot{x}_1 = -ax_1$ converges exponetially fast. (ii) Once reached, the manifold cannot be left. (iii) Robust to parameter variation.

Chattering: Delays in system or controller can cause chattering.

Chattering reduction by Separation: Separate the continuous and switching components. Use nominal $\hat{h}$, $\hat{g}$ in the control law

$$u = -\left\| \frac{a_1 x_2 + \hat{h}(x_1, x_2)}{\hat{g}(x_1, x_2)} \right\| - \beta(x_1, x_2) \operatorname{sign} s,$$

where $\beta(x_1, x_2) > \rho(x_1, x_2) + \beta_0$ for some $\beta_0 > 0$, if

$$\left\| \frac{\delta(x_1, x_2)}{g(x_1, x_2)} \right\| \leq \rho(x_1, x_2)$$

with $\delta(x_1, x_2) = a_1(1 - g(x_1, x_2)/\hat{g}(x_1, x_2))x_2 + (h(x_1, x_2) - \hat{h}(x_1, x_2)g(x_1, x_2)/\hat{g}(x_1, x_2))$.

Chattering reduction of high gain saturation function: Use the control law with saturation

$$u = -\beta(x_1, x_2) \operatorname{sat}\left(\frac{s}{\varepsilon}\right),$$

and a Lyopanov function $V = 0.5 x_1^2$, we can show ultimate boundedness so that all trajectories reach the (invariant) set

$$\Omega_\varepsilon = \left\{ x_1 \leq \frac{\varepsilon}{a_1}, s \leq \varepsilon \right\},$$

in finite time. If ε decreases, the ultimate bound T and chattering increases $\rightarrow$ trade-off stability vs performance.

Generalization: We can generalize using the system $y^{(n)} = h(\mathbf{x}) + g(\mathbf{x})u$ with $\mathbf{x} = [y, \dot{y}, \ldots, y^{(n-1)}]$. Assume bounds $\|h(\mathbf{x})\| < \varepsilon(\mathbf{x})$, $\|g(\mathbf{x})\| < \mu(\mathbf{x})$. We use this for tracking a trajectory $\mathbf{x}_d(t)$, i.e., we have the manifold

$$s(\mathbf{x}) = e_y^{(n-1)} + a_1 e_y^{(n-2)} + \ldots + a_{n-1} e_y = 0$$

using $e = y - y_d$. For $s(\mathbf{x}) = (d/dt + \lambda)^{n-1} e_y$, we can determine whether the a_i's are suitable by checking $z^{n-1} + a_1 z^{n-2} + \ldots + a_{n-1} = 0$ is Hurwitz.

8.5 Lyopanov Redesign

System: The system with perturbation δ is given by $\dot{x} = f(x, u, t) + g(x, t)[u + \delta(x, u, t)]$. Assume, you have a controller $u = \psi(x, t)$, which stabilizes the nominal system (i.e., $\delta(x, u, t) = 0$).

Lyopanov function: Determine a Lyopanov function so that

$$\alpha_1(\|\mathbf{x}\|) \leq V(\mathbf{x}, t) \leq \alpha_2(\|\mathbf{x}\|), \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}}^T [f(\mathbf{x}, t) + g(x, t)\psi(x, t)] \leq -\alpha_3(\|\mathbf{x}\|).$$

Perturbation Assumption: Assume that for $u = \psi(x, t) + v$, we have the bound

$$\|\delta(x, \psi(x, t) + v, t)\| \leq \rho(x, t) + \varkappa_0 \|v\|,$$

with $0 < \varkappa_0 < 1$. Nota bene: Only bounds on perturbation, $\rho(x, t)$ can be large but has to be known.

Design of v: Using $\dot{V} \leq -\alpha_3\left(\|x\|\right) + w^T v + w^T \delta$ with $w = (\partial V/\partial x) g(x,t)$, we realize that $w^T v + w^T \delta \leq 0$.

1. **Norm 2:** This is achieved by

$$v = -\eta(x,t) \frac{w}{\|w\|}$$

   with $\eta(x,t) \geq \rho(x,t)/(1-\varkappa_0)$. This gives us the control law

$$u = \psi(x,t) - \frac{\rho(x,t)}{1-\varkappa_0} \frac{w}{\|w\|},$$

   which is stable.

2. **Norm ∞:** This is achieved by

$$v = -\eta(x,t) \operatorname{sign} w$$

   with $\eta(x,t) \geq \rho(x,t)/(1-\varkappa_0)$.

3. **Norm 1:** This is achieved by

$$v = -\eta(x,t) \frac{w}{\|w\|}$$

   with $\eta(x,t) = \rho(x,t)/(1-\varkappa_0)$.

8.6 Backstepping

To be continued...