Lecture Notes 1

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1 Basic Differential Equations

We all know that the differential equation $\dot{x} = Ax$ has the unique solution $x(t) = x_0 \exp(-At)$. This, however, does not mean that all differential equations have solution.

Example 1 *We have*

$$\dot{x} = x^2. \tag{1}$$

When defining $x = 1/y$ *and differentiating* $\dot{x} = -\dot{y}/y^2$. *Thus, we obtain*

$$-\frac{\dot{y}}{y^2} = \dot{x} = x^2 = \left(\frac{1}{y}\right)^2, \tag{2}$$

which implies $\dot{y} = -1$. *By integration, we see that* $y(t) = \int_0^t \dot{y} dt = -t + y_0$. *When inserting* $y(t) = 1/x(t)$, *we obtain*

$$x(t) = \frac{x_0}{1 - tx_0}. \tag{3}$$

It is clear that the solution of $x(t)$ *does not exist at* $t = 1/x_0$, *i.e., it has* ***no global solution***.

This, we achieve by studying the whether we have a fixpoint.

1.1 Fixpoint Problem

We intend to test whether differential equations of the kind

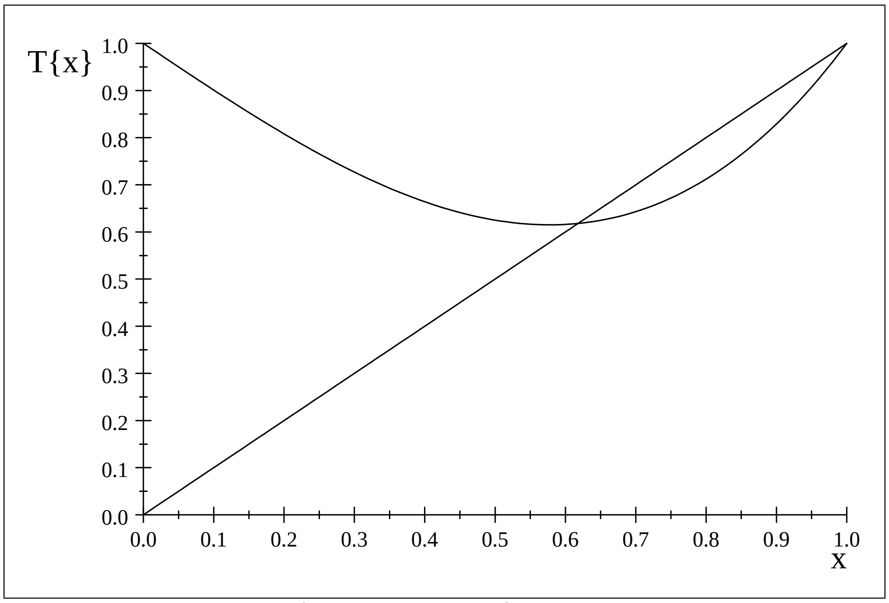
$$\dot{x} = f(x) \tag{4}$$

have a solution. This is equivalent to the Volterra integral equation

$$x(t) = x(0) + \int_0^t f(x(\tau))d\tau = T\left\{x(\cdot)\right\}, \tag{5}$$

where $T : \mathcal{C}^0\left(\mathbb{R}^+\right) \to \mathcal{C}^0\left(\mathbb{R}^+\right)$ is an operator. Thus, there exists a solution if $x\left(\cdot\right) = T\left\{x\left(\cdot\right)\right\}$ has a solution.

Example 2 *For $T : [0,1] \to [0,1]$ being a continous function, we have a fixed point at x_* if x and $T\{x\}$ intersect at $x = x_*$. For example, for $T\{x\} = x^3 - x + 1$ has one solution in $[0,1]$ at $x_* = 0.618\,03$ and $x_* = 1.0$. See Figure 1.1.*



This function has two fixed points.

Any continuous function from $[0,1]$ onto $[0,1]$, will have one intersection with itsself. Thus, it would be a fixed point.

Theorem 3 (Brouwer Fixed Point) *Any continuous function $f : \mathcal{C} \to \mathcal{C}$ from a convex, compact set to itself has at least one fixed point.*

How can we determine such a fixed point?

1.2 Contraction Mapping Theorem

Assume that we have $x(\cdot) = T\{x(\cdot)\}$, and a solution $x^\infty(\cdot)$. For such a solution, we have

$$x^\infty(\cdot) = T\{x^\infty(\cdot)\} = TT\{x^\infty(\cdot)\} = T^n\{x^\infty(\cdot)\}. \tag{6}$$

Can this property be used for determining the solution $x^\infty(\cdot)$ using an initial guess $x^0(\cdot)$?

Example 4 *We apply the operator*

$$T^n\left\{x^0(\cdot)\right\} \tag{7}$$

with $n \to \infty$ on arbitrary $x^0(\cdot)$. For (i) $T\{x\} = 0.25 + 0.6x$ and (ii) $T\{x\} = 0.5(1 - \cos(x\pi))$, this is illustrated in Figure 1. We can observe that "if the

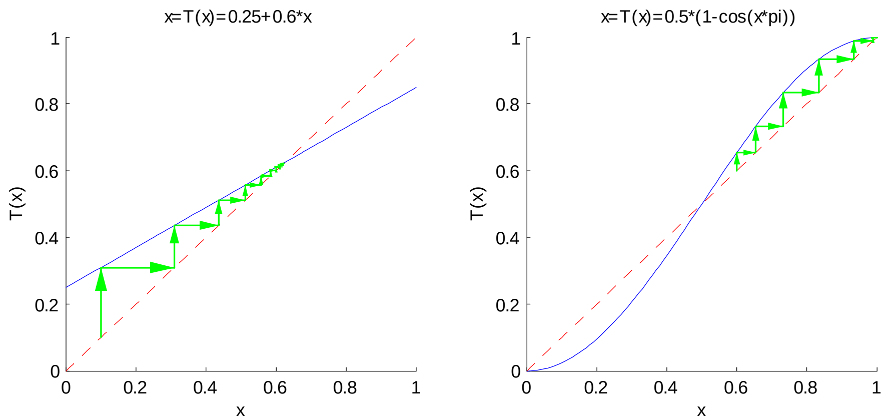


Figure 1: This figure illustrates the contraction theorem. Note that if $T\{x\} > x$, it will go up and for $T\{x\} < x$ down. For $T\{x\} = x$, we have a fixed point.

slope is less me", it will converge to a fixed point otherwise it will diverge. Check out `fixedPoint.m`*.*

The example illustrates the problem but we need to formalize it. First, we need the definition of a contraction.

Definition 5 *T is a contraction if there exists a $\rho \in [0,1)$ such that*

$$\|T\{x(\cdot)\} - T\{y(\cdot)\}\|_\infty \le \rho \|x(\cdot) - y(\cdot)\|_\infty, \tag{8}$$

where $\|e(\cdot)\|_\infty \equiv \max_t \|e(t)\|_2 = \max_t \sqrt[2]{x_1^2 + \ldots + x_n^2}$.[1]

This definition can be translated that if the maximal error $\|x(\cdot) - y(\cdot)\|_\infty$ over all time t gives us a bound on the maximal slope $\|T\{x(\cdot)\} - T\{y(\cdot)\}\|_\infty$, then we will call T a contraction.

Theorem 6 *If T is a contraction, then the repeated iteration of T, i.e.,*

$$x^\infty(\cdot) = \lim_{n\to\infty} T^n\left\{x^0(\cdot)\right\} \tag{9}$$

converges to a unique fixed point.

Proof. Ever Cauchy sequence $x_1, x_2, x_3, \ldots, x_n$ converges if $\lim_{n,m\to\infty} |x_n - x_m| = 0$. Thus, we have to show that ■

The next question is whether T can be turned into a contraction.

[1] Any assignment of positive numbers to vectors is a norm if (i) $\|0\| = 0$, (ii) $\forall x \neq 0.\ \|x\| > 0$, (iii) $\|x + y\| \le \|x\| + \|y\|$. Norms can also be written as $\|x\| = (x, x)$.

Definition 7 *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz if there exists an ι such that*

$$\|f(x) - f(y)\|_2 \leq \iota \|x - y\|_2 \,. \tag{10}$$

We realize that we always have for small δt, the difference $f\left(x\left(\tau\right)\right) - f\left(y\left(\tau\right)\right)$ remains constant, and therefore

$$\begin{aligned} \left\|T\left\{x\left(\cdot\right)\right\} - T\left\{y\left(\cdot\right)\right\}\right\|_\infty &= \max_{\delta t} \left\| \int_0^{\delta t} f\left(x\left(\tau\right)\right) - f\left(y\left(\tau\right)\right) d\tau \right\|_2, &(11)\\ &\approx \left\| \int_0^{\delta t} f\left(x\left(\tau\right)\right) - f\left(y\left(\tau\right)\right) d\tau \right\|_2, &(12)\\ &= \delta t \left\| f\left(x\left(t\right)\right) - f\left(y\left(t\right)\right)\right\|_2. &(13)\end{aligned}$$

This results into the theorem below for $\delta t < 1$.

Theorem 8 *If f is Lippschitz and if the integration interval is small enough, T is a contraction.*

We illustrate this in a few examples.

Example 9 *Some examples to illustrate the application of this theorem.*

1. *We have $\dot{x} = ax$, thus we have*

$$\begin{aligned} T\left\{x^0\right\} &= x_0 + \int_0^t f\left(x^0\left(\tau\right)\right) d\tau = x_0 + \int_0^t ax \; d\tau &(14)\\ &= x_0 + \left[ax_0\tau\right]_0^t = x_0\left(1 + at\right) = x^1, &(15)\\ T\left\{T\left\{x^0\right\}\right\} &= T\left\{x^1\right\} = x_0 + \int_0^t a\left(x_0\left(1 + at\right)\right) \; d\tau &(16)\\ &= x_0\left(1 + at + \frac{1}{2}a^2t^2\right), &(17)\\ T^n\left\{x^0\right\} &= x_0\left(\sum_{n=0}^{n} \frac{a^n}{n!}t^n\right) = x_0 \exp\left(at\right). &(18)\end{aligned}$$

This solution might look oddly familiar. We see that

$$\|f(x) - f(y)\|_2 = a\|x - y\|_2 \leq \iota \|x - y\|_2 \,, \tag{19}$$

and thus for $a \leq \iota$, there will be a solution.

2. *For $\dot{x} = tx$, we have*

$$T\left\{x^0\right\} = x_0 + \int_0^t f\left(x^0\left(\tau\right)\right) d\tau = x_0 + \int_0^t \tau x \; d\tau \tag{20}$$

$$= x_0 \left(1 + \frac{t^2}{2}\right) = x^1, \tag{21}$$

$$T\left\{T\left\{x^0\right\}\right\} = T\left\{x^1\right\} = x_0\left(1 + \frac{t^2}{2} + \frac{t^4}{8}\right), \tag{22}$$

$$T^n\left\{x^0\right\} = x_0\left(\sum_{n=0}^{n} \frac{1}{n!}\left(\frac{t^2}{2}\right)^n\right) = x_0 \exp\left(\frac{1}{2}t^2\right). \tag{23}$$

This solution might look oddly familiar. We see that

$$\left\|f(x) - f(y)\right\|_2 = t\left\|x - y\right\|_2 \le \iota\left\|x - y\right\|_2, \tag{24}$$

and thus for $t \le \iota$, there will be a solution.

3. *For $\dot{x} = x^2$, we look at*

$$x^2 - y^2 = (x - y)(x + y), \tag{25}$$

and, thus, f will be Lippschitz for $\left\|x + y\right\|_2 \le \iota$, as

$$\left\|f(x) - f(y)\right\|_2 = \left\|x - y\right\|_2 \left\|x + y\right\|_2 \le \iota\left\|x - y\right\|_2, \tag{26}$$

will be valid then. Let us assume $x, y \in [-\iota/2, \iota/2]$, and we had an initial point $x_0 = \iota/4$.Then, the minimal speed will be $\dot{x}_0 = x_0^2 = (\iota/2)^2$ and it will have travelled at least until $x_{\max} = x_0 + f\left(x_0\right)\delta t = \iota/4 + (\iota/2)^2\,\delta t$ after δt. The state x will reach $\iota/2$ at

$$\frac{\iota}{2} = \frac{\iota}{4} + \frac{\iota^2}{4}\delta t \Longleftrightarrow \delta t < \frac{1}{\iota}. \tag{27}$$

Thus, it does not guarantee the existence of a solution until δt. Similar arguments can be made for different initial values $x_0 > 0$.