About a novel general underlying principle[*] of mechanics.

The orignal paper by Carl Friedrich Gauss,
translated by Jan Peters.

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It is well-known that the principle of virtual velocities[1] transforms the analysis of static systems into a mathematical excercise; through D'Alemberts principle, dynamic systems analysis can again be reduced to static systems analysis. It is therefore natural that there cannot exist any novel underlying principle in the study of motion or equilibriae; every new principle is either contained in the two previously mentioned principles or it can be derived from them. However, it has become apparant that this does not imply that every new principle is worthless. It will always be both interesting and instructive to gain new, advantageous points of view as it might help to solve one or another problem more efficiently or more appropriately. The grand surveyor[2] – who has constructed the "house of mechanics" in such a brilliant way using the principle of virtual velocities – would not have despised it. We intend to lift Maupertuis' principle of the minimum impact up to its greater destiny and generality[3] as it can be advantagous in many cases[4].

[*] Literally "Grundgesetz" means basic law; however, its meaning is more like underlying principle.

[1] He says virtual velocities instead of virtual displacements!

[2] He refers to D'Alambert, I believe.

[3] This sentence was strange as it was missing the "we intend".

[4] **Footnote by Gauss himself:** I allow myself to criticize another surveyor. I believe that the way how HUYGENS' law for double light fraction in glass is being proved – using the principle of the minimum impact – is insufficient. Indeed, the applicability of this principle depends on the conservation of "living" forces. According to this, velocities of point masses can only depend on the location - without the direction having any influence. However, this is required in the described experiment. It appears to me as if all attempts to treat double fraction with the laws of dynamics are prone to fail unless we treat the light particles as pure points.

The strange character of the principle of virtual velocities is that it is a general equation for solving all static problems and therefore it is a proxy of all other principles, plausible by itself. This happens without the reasons being directly apparent.

In *this* way, the novel principle which I intend to present is advantagous. Furthermore, it has the second advantage that it handles both motion and rest in the same, more general manner. It is very much in order that during the education in sciences and the teaching of a person, that the simpler topic preceeds the more difficult one, the straight-forward topic preceeds the entangled one, and the special topic preceeds the general one. However, once a human being has mastered the higher level, this relationship reverses and statics appear to be just a special case of mechanics. Even the surveyor mentioned above appears to value this point of view as he acknowledges the virtue of the principle of minimum impact that it can handle both equilibrium and motion – if one expresses it in a way the the "living"[5] forces are the smallest for both. This comment appears somehow more amusing than true as such a minimum in both cases has completely separate meanings.

The novel principle states the following:

The motion of a system of point masses – connected with each other in an arbitrary way and bound by arbitrary external constraints – happens at any point in time with largest possible conformance with the unconstrained motion, or under the minimal possible constraint[6]*. In here, the measure of the constraint of the system of particles at any time is given by the sum of the products of the squares of the deviation of each particle from the free motion weighted by its mass.*

We denote by m, m', m'', $\ldots$ the point masses and by a, a', a'', $\ldots$[7] their locations at time t. By b, b', b'', $\ldots$ we denote the locations at which the particle masses would be after an infinitisimal amount[8] of time dt – due to the forces applied upon the particles and due to the velocity and direction if they were free.

[5]By "living" he most likely means non-virtual?

[6]Gauss uses "Zwang" which can be translated both a constraint as well as force. Maybe I should change this to constraint forces?

[7]Gauss uses u.s.w. which is equivalent to "et cetera" or etc; for making it easier, I chose "$\ldots$".

[8]Gauss using an amusing word for this time increment, he calls it a "time-particle".

The real locations c, c', c'', ... will be the ones for which $m(bc)^2+m'(b'c')^2+m''(b''c'')^2 \dots$ has a minimum while fulfilling all conditions upon the system.

Apparantly, an equilibrium is just a special case of this general law, and the condition for it is that

$$m(ab)^2 + m'(a'b')^2 + m''(a''b'')^2 + \dots$$

itself is at a minimum. Put in a different way, the maintainance of the current state of the system is closer to the free motion of the single particles than any change of state.

The derivation of this principle from the one mentioned above is straightforward when accomplished in the following way.

The force acting upon a point mass m consists apparently out of two components. Firstly, there is one component which would lead the particle from a to c in the time dt given the initial velocity and direction[9] at time t. There is second component which would move the particle from rest in c along the line $\overline{cb}$ if the particle was free. The same can be said about all the other particles. According to D'Alamberts principle, the point masses m, m', m'', ... have to move alone under the application the second force component to $\overline{cb}$, $\overline{c'b'}$, $\overline{c''b''}$, ... while the conditions of the system ensure equilibrium at the places c, c', c'', ... [10].

In accordance with the principle of the virtual velocities, this requires an equilibrium so that the sums of the products, (i.e., the the masses m, m', m'', ..., the lines $\overline{cb}$, $\overline{c'b'}$, $\overline{c''b''}$, ... and arbitrary others projected onto the lines[11]) allow the motion of these points under the conditions of the system to be always =0, as one would normally say[12],

[9] Apparantly for Gauss, velocity is just the magnitude of the velocity and direction is the direction of velocity.

[10] This sentence of Gauss is near incomprehensible in German.

[11] I am not sure how much sense this makes; but Gauss is very lax in his wording here.

[12] **Footnote by Gauss himself:** The normal expression requires without mentioning that for every possible motion, the motion into the opposite direction is also possible. E.g., that just like a point mass is required to stay on a plane, also the distances between two points remains the same, etc. This alone is an uneccesary limitation, often not corresponding to reality. The surface of an impermeable body requires a body not to stay on top of it but only restricts it from entering at one side; a tense, not expandable but bendable thread between two points allows the distance to decrease but not increase, etc. Why should we express the law of virtual velocities without including these cases from the beginning on?

or more correctly, that the sum could never become positive. Let us assume γ, γ', γ'', … are locations different from c, c', c'', … but fulfilling the conditions of the system. Furthermore, let θ, θ', θ'', … be the angles formed between the lines $\overline{c\gamma}$, $\overline{c'\gamma'}$, $\overline{c''\gamma''}$, … and $\overline{cb}$, $\overline{c'b'}$, $\overline{c''b''}$, …. Then $\sum m \cdot \overline{cb} \cdot \overline{c\gamma} \cos\theta$ is either 0 or negative. Since

$$\overline{\gamma b}^2 = \overline{cb}^2 + \overline{c\gamma}^2 - 2\overline{cb} \cdot \overline{c\gamma} \cdot \cos\theta,$$

it becomes clear, that

$$\sum m \cdot \overline{\gamma b}^2 - \sum m \cdot \overline{cb}^2 = \sum m \cdot \overline{c\gamma}^2 - 2\sum m \cdot \overline{cb} \cdot \overline{c\gamma} \cdot \cos\theta$$

is also always positive. Therefore $\sum m \cdot \overline{\gamma b}^2$ will always be bigger than $\sum m \cdot \overline{cb}^2$, which implies that $\sum m \cdot \overline{cb}^2$ is a minimum. q.e.d.

It is strange that free motion – if it cannot coexist with the required constraints – will be modified by Nature in exactly the same manner as the experienced mathematician will balance the experiences under the influence of different connected variables using the least-squares method. This analogy can be continued but that is not my goal.