Trajectory Planning with Adaptive Probabilistic Models

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I. Problem Formulation

Consider a control system with state $x \in \mathcal{X}$, inputs $u \in \mathcal{U}$ and dynamics described by an ordinary differential equation subject to inequality constraints:

$$\dot{x}(t) = f(x(t), u(t)), \qquad F(x(t)) \geq 0. \tag{1}$$

A complete trajectory of the system is written as $\pi : [0, T] \to \mathcal{X} \times \mathcal{U}$ for some finite time $T > 0$ so that $\pi(t) = (x(t), u(t))$ encodes both the state and controls at time t. The goal is to compute the optimal control signal $u^*(t)$ driving the system from its initial state $x_0 \in \mathcal{X}$ to a given goal region $\mathcal{X}_g \subset \mathcal{X}$ while minimizing a given performance measure. Let $\mathcal{P}$ denote the space of all feasible trajectories satisfying the dynamics, constraints, and boundary conditions. The objective is to compute

$$\pi^* = \underset{\pi \in \mathcal{P}}{\operatorname{argmin}}\, J(\pi), \qquad \text{where } J(\pi) = \int_0^{\tau(\pi)} C(\pi(t))\mathrm{d}t,$$

where $\tau(\pi)$ gives the trajectory time duration and C is a given cost typically encoding time and control effort, e.g. $C(x(t), u(t)) = 1 + \lambda \|u(t)\|^2$ with a chosen weight $\lambda \geq 0$.

II. Dynamical System Representations

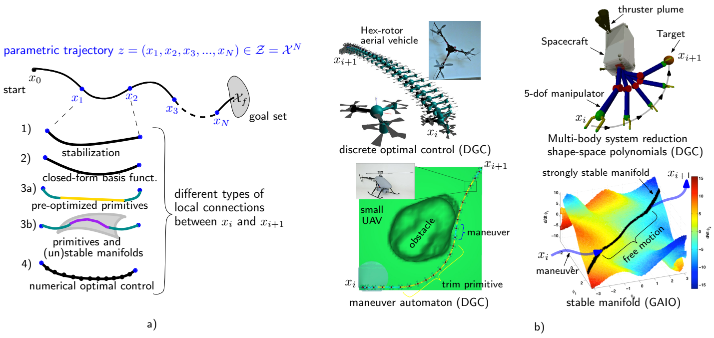

Fig. 1. a) various methods for local trajectory generation: starting with simple but suboptimal stabilization, including methods exploiting dynamical structures, to most general but least efficient numerical optimal control; b) examples of such techniques that we have constructed in the context of autonomous vehicle planning and control.

Each trajectory is parametrized using N number of "way-points" and a mapping $\varphi : \mathcal{Z} \to \mathcal{P}$ reconstructing the continuous trajectory π, i.e.

$$z = (x_1, ..., x_N) \in \mathcal{Z} = \mathcal{X}^N \quad \Leftrightarrow \quad \pi(t) = \varphi(z, t)$$

The mapping implicitly encodes a local dynamically feasible connection method between states x_i and x_{i+1}. Thus, a given parameter z corresponds to a unique trajectory composed of local connections between (x_0, x_1), (x_1, x_2), ..., and $(x_N, \mathcal{X}_g)$.

III. Probabilistic Trajectory Optimization

Our approach employs an *importance density* $q(Z)$ over the space of parametrized trajectories and adapts the density online until its mass becomes concentrated around the approximately optimal trajectory $z^* = \arg\min J(z)$. This is accomplished by computing the probabilities:

$\mathbb{P}(J(Z) \leq \gamma)$: cost of a trajectory is less than γ,
$\mathbb{P}(F(Z) \geq 0)$: trajectory is feasible,

iteratively while automatically lowering the cost γ until convergence.

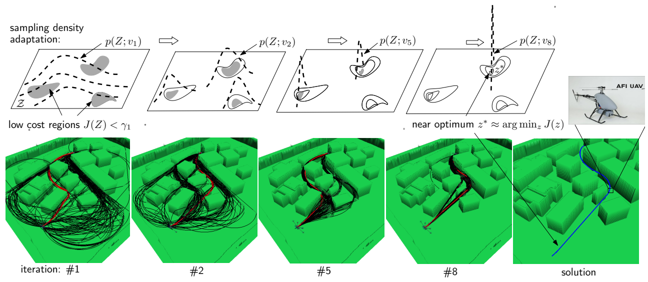

Fig. 2. Randomized trajectory optimization using an adaptive distribution that automatically focuses in high-performance regions. The task is to compute a time-optimal obstacle-free trajectory for a helicopter modeled as a non-trivial underactuated systems in 3-D.

A. Optimization through Density Estimation

The first approach is to compute $q(z)$ directly through the minimization $\min_q \mathrm{KL}(q^* \,\|\, q)$, where

$$q^*(z) = \frac{I_{\{J(z) \leq \gamma \wedge F(z) \geq 0\}} p(z)}{\mathbb{P}(J(Z) \leq \gamma) \cdot \mathbb{P}(F(Z) \geq 0)}, \tag{2}$$

where $p(Z)$ is some base measure on $\mathcal{Z}$ that for instance can incorporate prior knowledge about desirable trajectories. In computational convenience and efficiency we assume a parametric distribution $q(z) = p(z; v)$ where $v \in \mathcal{V}$ is the parameter. Problem (2) is solved approximately by finding the optimal parameter v^* according to $\hat{v}^* = \operatorname{argmax}_{v \in \mathcal{V}} \frac{1}{N} \sum_{i=1}^{N} I_{\{J(Z_i) \leq \gamma \wedge F(Z_i) \geq 0\}} \log p(Z_i, v)$, where $Z_1, ..., Z_n$ are i.i.d. samples from a base measure $p(\cdot, v_0)$.

B. Optimization through Function Approximation

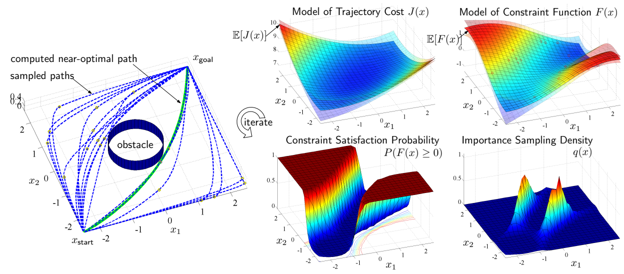

Fig. 3. A simple optimal planning problem solved using Gaussian Process models of the cost $J(x)$ and constraints $F(x)$. The plots show the evolved models after 20 iterations. Remarkably, the importance density q clearly indicates that the optimal region to select x are the states around the border of the obstacle that are reachable from both start and goal.

The second approach is to construct probabilistic models of the functions $J(z)$ and $F(z)$ in order to predict the performance of unobserved trajectories. In this case the probability density will be artificially constructed according to

$$q(z) \propto \mathbb{P}(J(z) < \gamma) \cdot \mathbb{P}(F(z) \geq 0), \tag{3}$$

where here J and F are regarded as random functions for each fixed parameter z. We will assume that the processes $J(z)$ and $F(z)$ have normal marginal distributions, i.e. they will be modeled as Gaussian Processes (GP). This is particularly convenient for constructing q in (3) through the simple expressions

$$\mathbb{P}(J(z) \leq \gamma) = \Phi\left(\frac{\gamma - \mathbb{E}[J(z)]}{\sqrt{\mathbb{V}[J(z)]}}\right), \quad \mathbb{P}(F(z) \geq 0) = \Phi\left(\frac{\mathbb{E}[F(z)]}{\sqrt{\mathbb{V}[F(z)]}}\right),$$

where $\Phi(\cdot)$ is the standard unit-normal CDF, and $\mathbb{E}[\cdot]$ and $\mathbb{V}[\cdot]$ denote expectation and variance. A simple example of a preliminary study is shown on Figure 3.