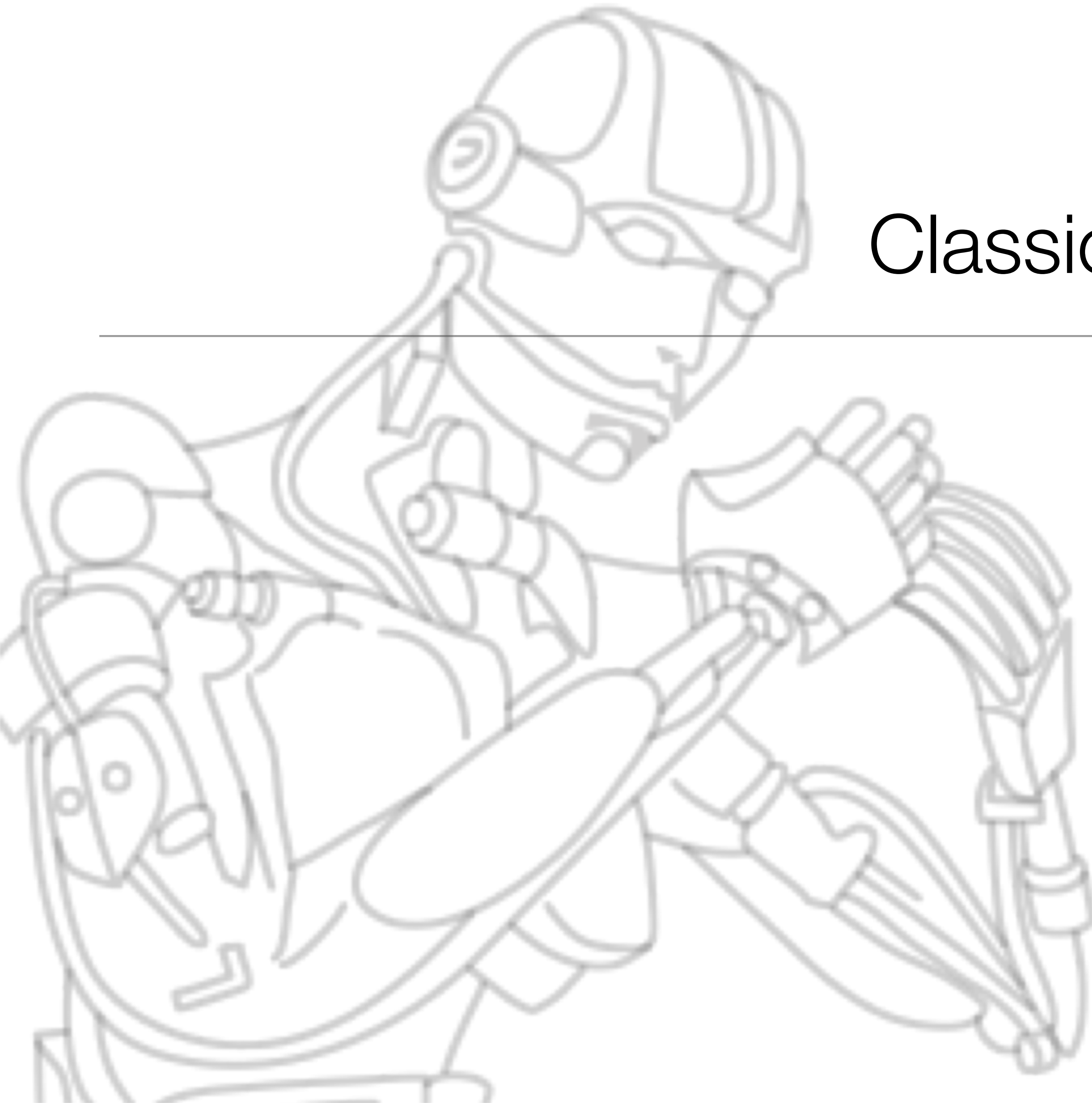


Classical Robotics in Nutshell

Jan Peters
Gerhard Neumann

Purpose of this Lecture

- What you need to know about robotics!
- Important robotics background in a nutshell!
- In order to understand robot learning, we have to understand the problems first

Content of this Lecture

1. What is a robot?

2. Modeling Robots
   - Kinematics
   - Dynamics

3. Representing Trajectories
   - Splines

4. Control in Joint Space
   - Linear Control
   - Model-based Control

5. Control in Task Space
   - Inverse Kinematics
   - Differential Inverse Kinematics

What is a Robot?

A robot is a reprogrammable multifunctional manipulator designed to move material, parts, tools, or specialized devices through variable programmed motions for the performance of a variety of tasks.

Robotics Institute of America

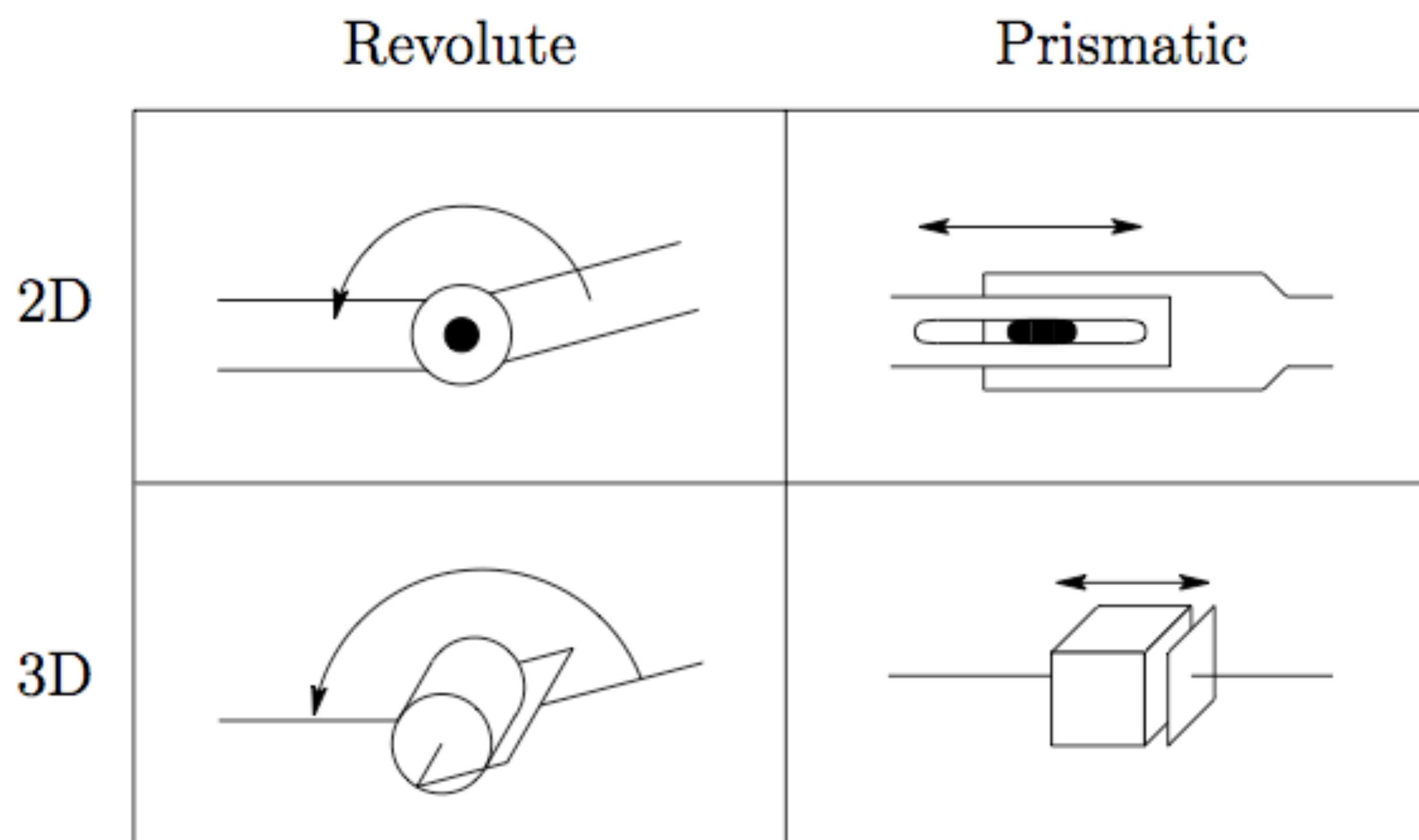
A computer is just amputee robot

G. Randlov

Modeling: What are the Degrees of Freedom?

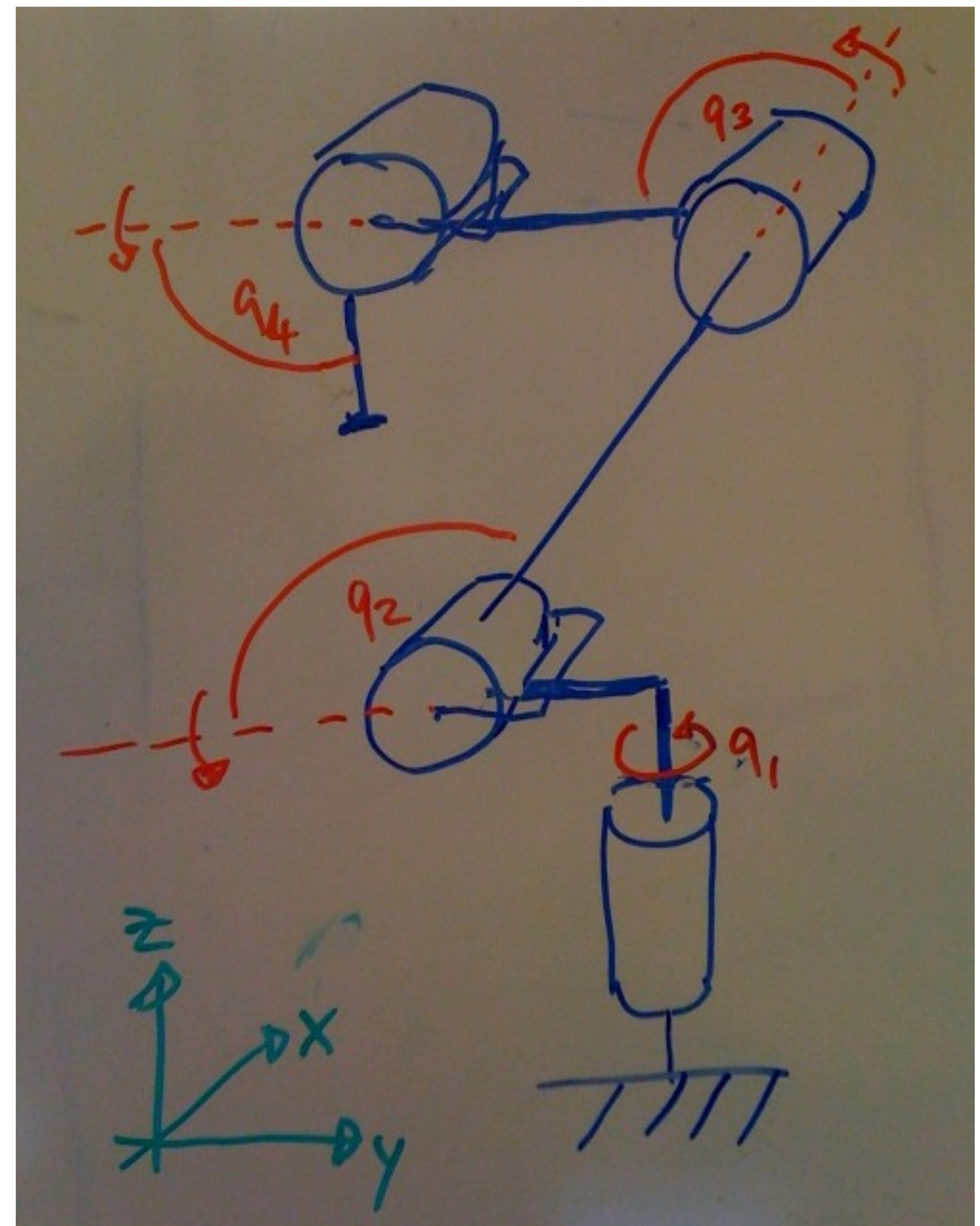
2 types of joints:

- revolute
- prismatic



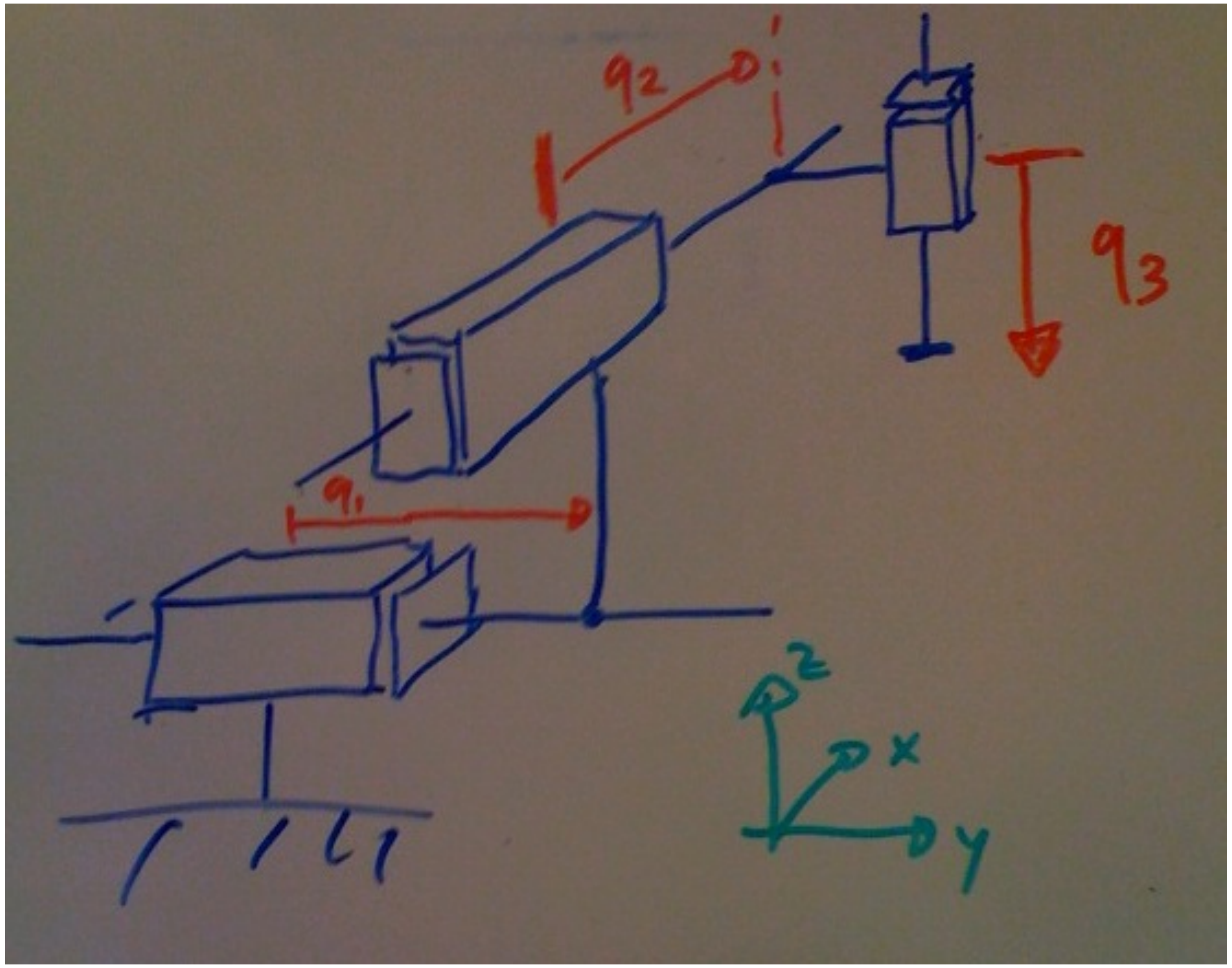
Modeling: What are the Degrees of Freedom?

Revolute joints



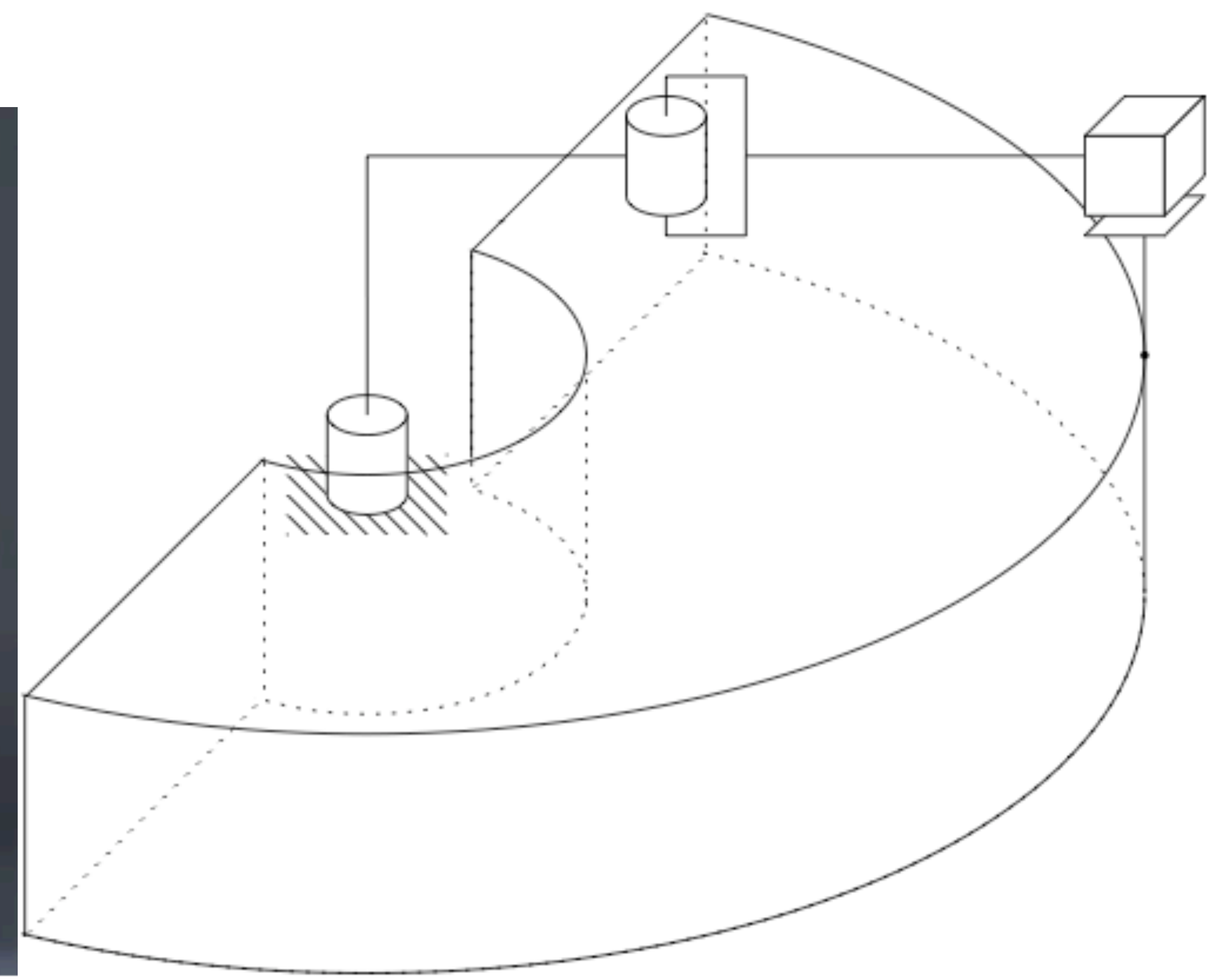
Modeling: What are the Degrees of Freedom?

Prismatic Joints



Workspace

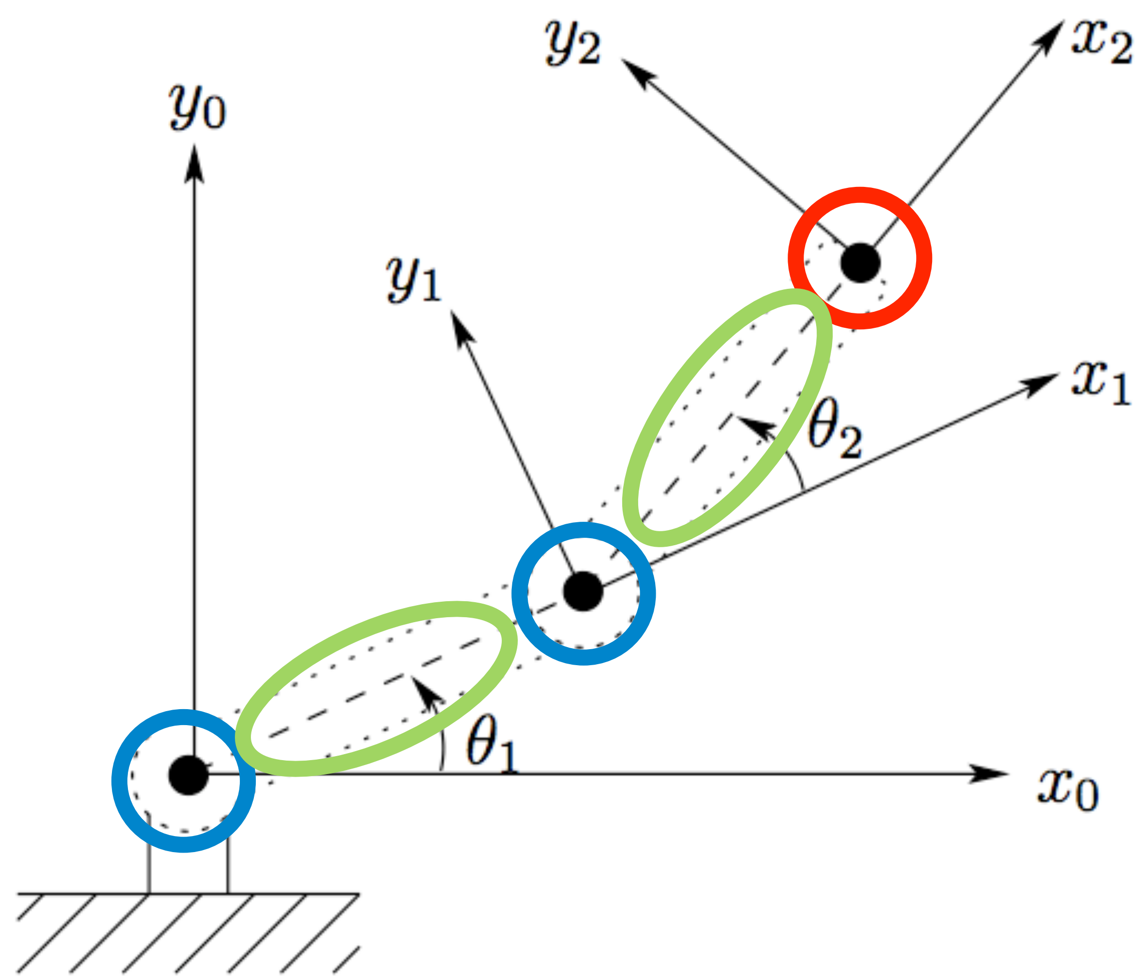
The workspace is the reachable space with the end-effector



Basic Terminology

Link

Joints: $\boldsymbol{q}$ [rad]



Task/Endeffector space: $\boldsymbol{x}$[m]

State (robot and environment): $\boldsymbol{s}$

Basic Terminology

Controls $\boldsymbol{u}$/ $\boldsymbol{a}$

- Velocities/Accelerations/Torques

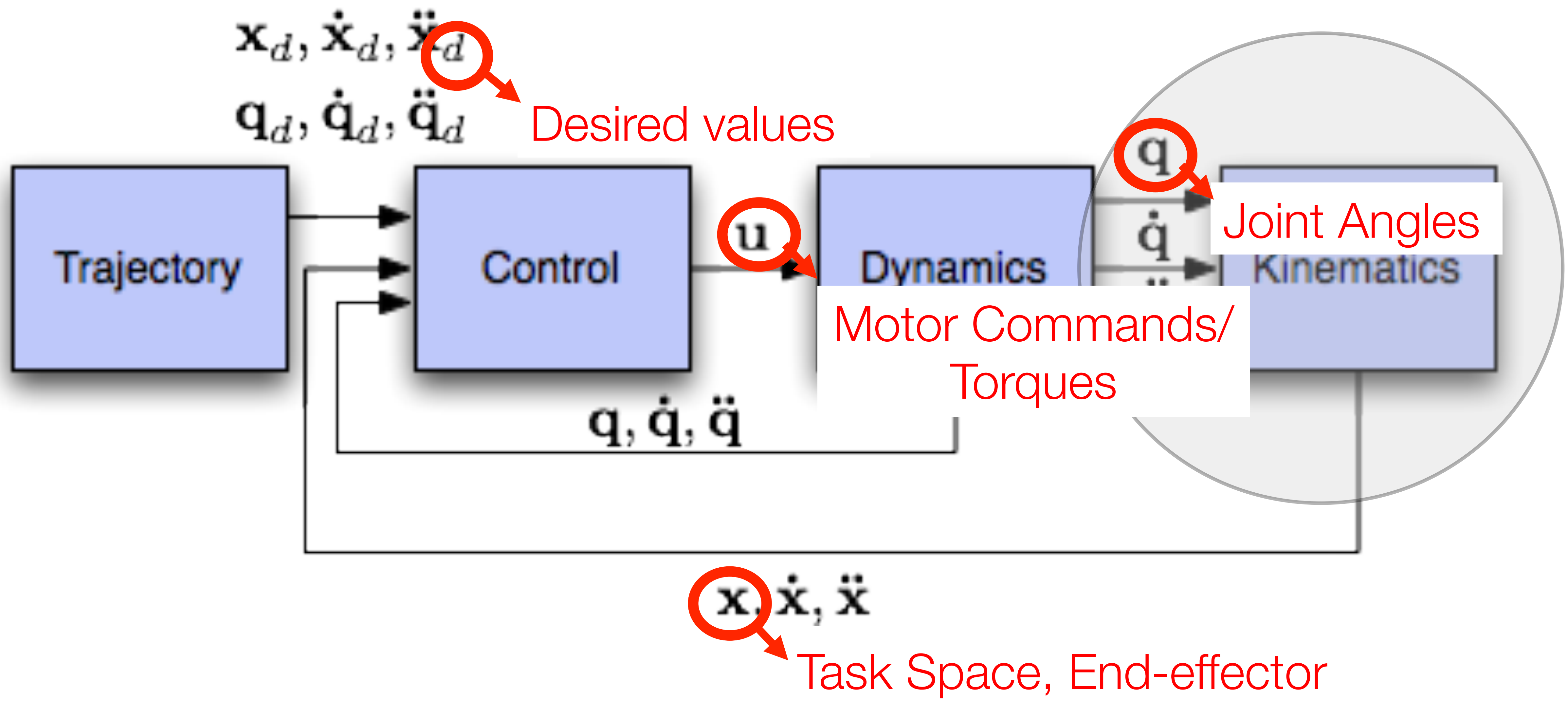
Torques: τ

Policy/Controller:

- Determinstic: $\boldsymbol{u} = \pi(\boldsymbol{s})$
- Stochastic: $\boldsymbol{u} \sim \pi(\boldsymbol{u}|\boldsymbol{s})$

Redundancy: #Joints > # Task Variables

Block Diagram of Complete System



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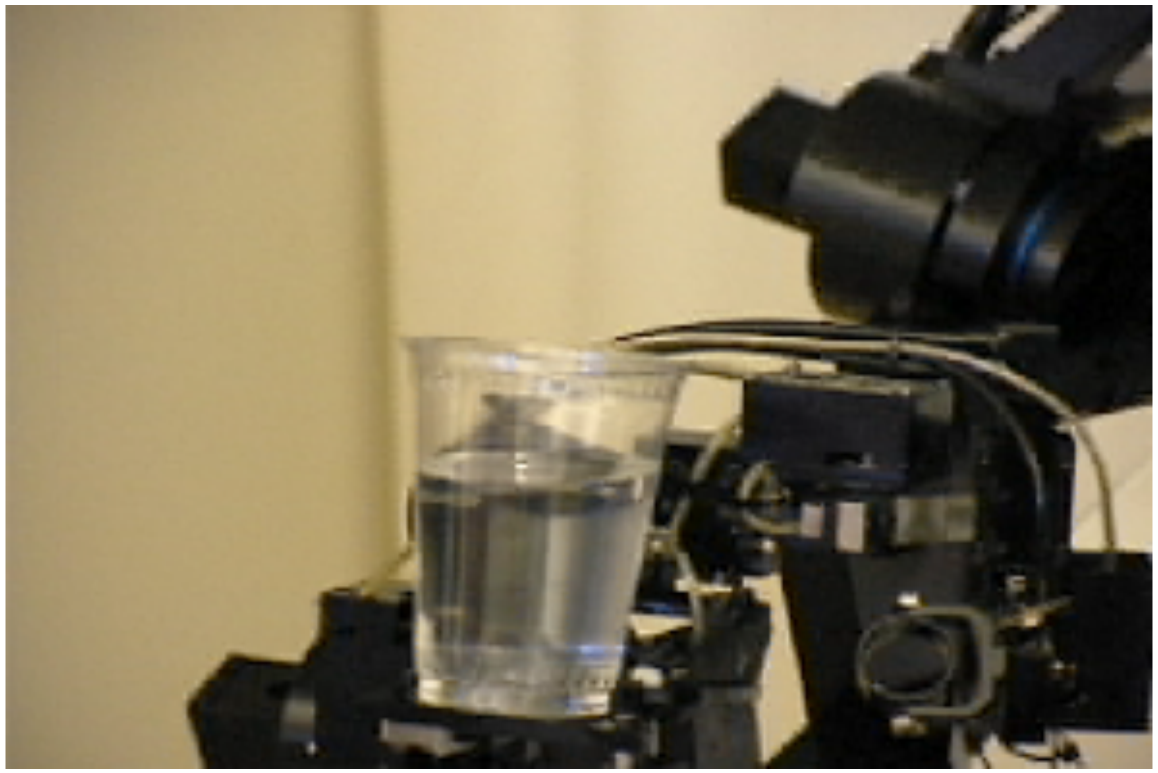
Kinematics



Where is my hand/endeffector
& what is it's orientation?

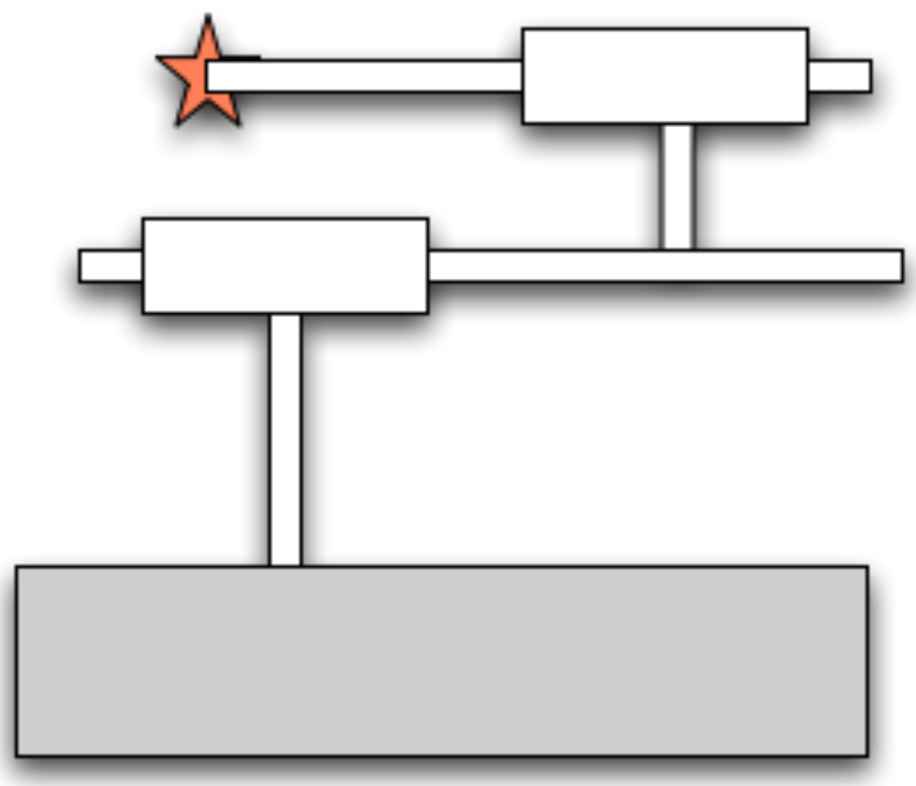
Where is my center
of gravity?

- **What do we want to have?**
- **Forward Kinematics:** A mapping from joint space to task space

$$\mathbf{x} = f(\mathbf{q})$$

Example 1: Prismatic Robot with 2 DoF

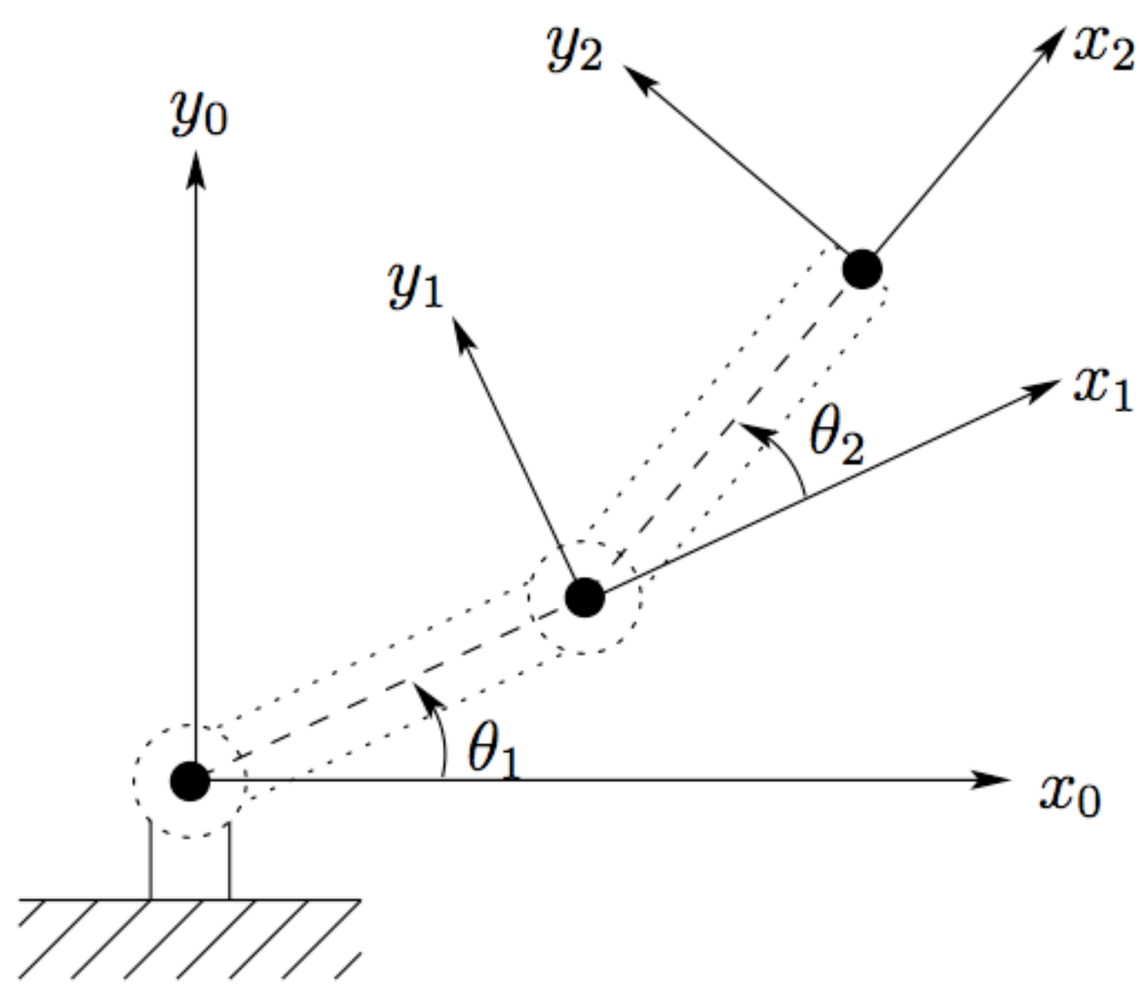
What are the forward kinematics $\mathbf{x} = f(\mathbf{q})$ **?**



$$x = q_1 + q_2$$

Example 2: Rotary Robot with 2 DoF

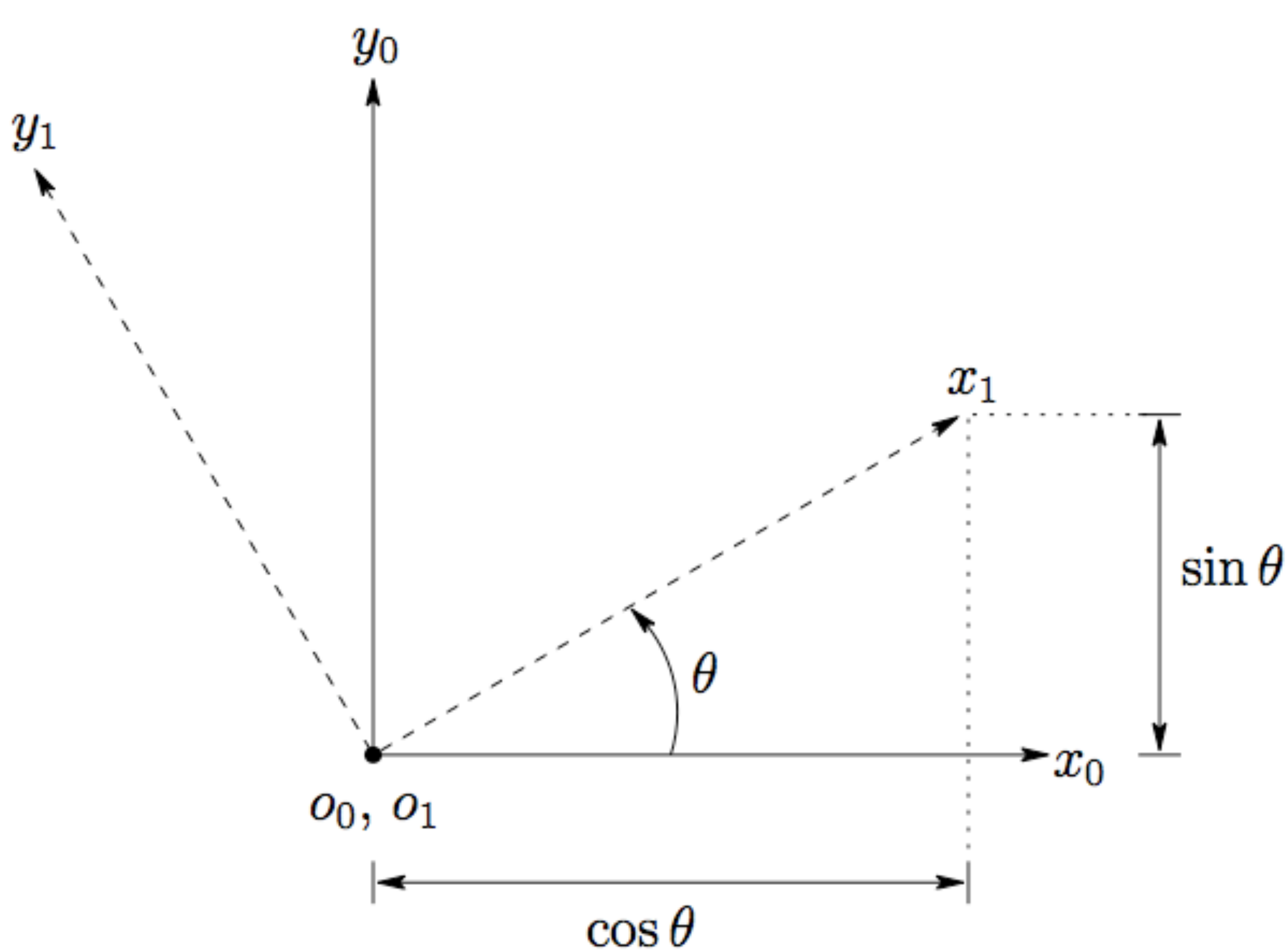
What are the forward kinematics $\mathbf{x} = f(\mathbf{q})$ **?**



$$
\begin{aligned}
x &= x_2 = a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\
y &= y_2 = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2)
\end{aligned}
$$

What does a "Rotation" mean?

- A rotation is a transformation of coordinate frames



Can we write the transformation as matrix multiplication?

- We want a matrix such that

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \mathbf{R}(\theta) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

- **Which matrix fulfills this?**
- We know that:

$$\mathbf{e}_x^1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \mathbf{R}(\theta)\mathbf{e}_x^0$$

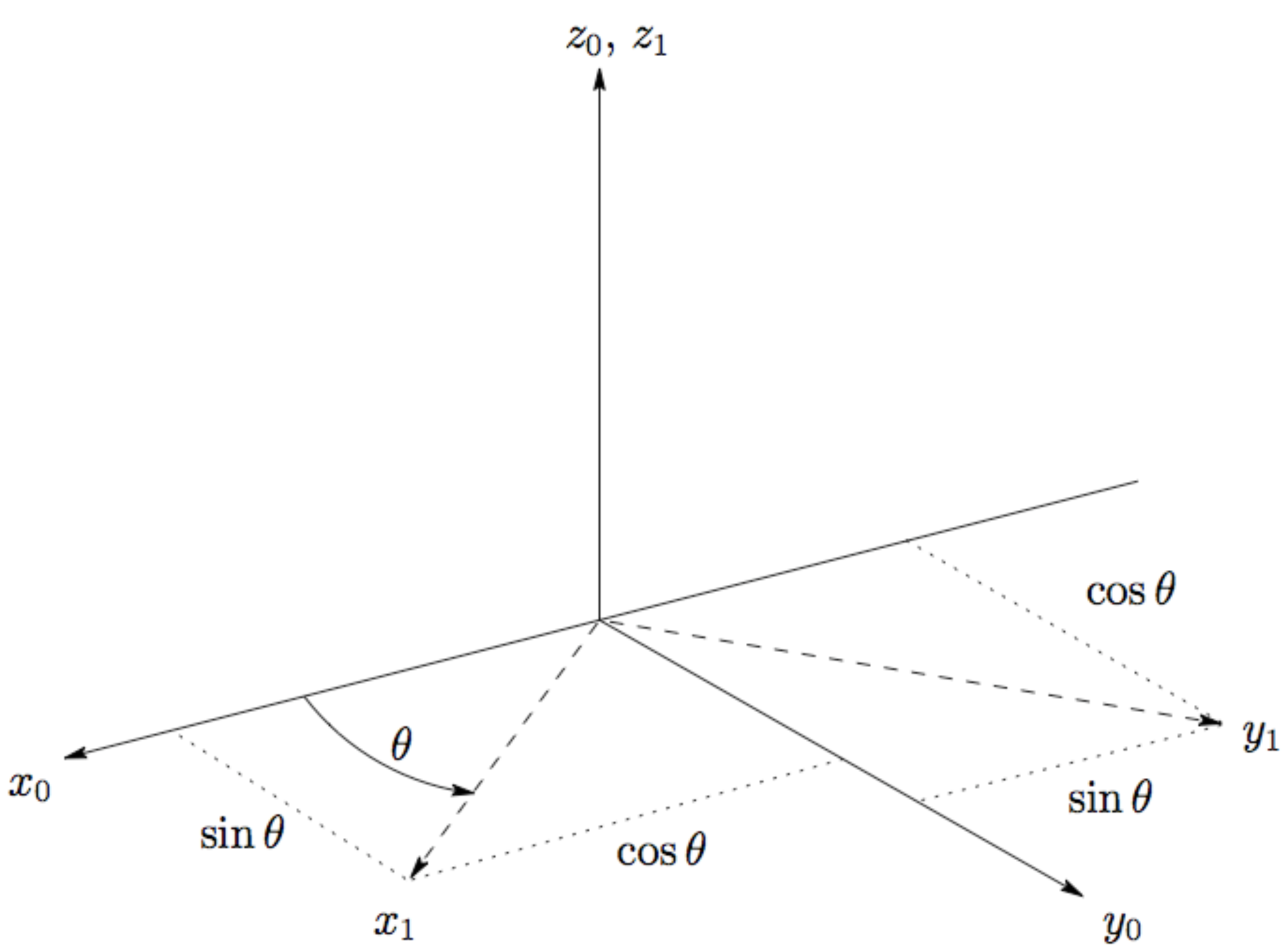
$$\mathbf{e}_y^1 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \mathbf{R}(\theta)\mathbf{e}_y^0$$

- Hence, we have

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotations in 3D

Rotations in 3D require rotating about any axis:



$$\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It's just like 2D, just add an identity for the axis around which you are rotating.

More about Rotations ...

Rotations can be stacked:

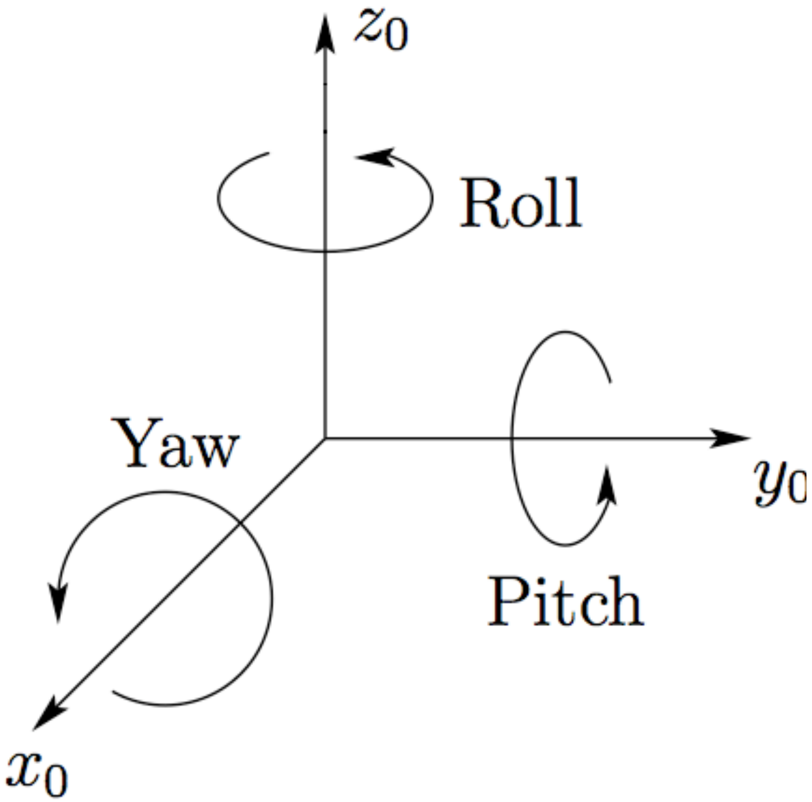
$$\begin{aligned} p^0 &= R_1^0 p^1 \\ p^1 &= R_2^1 p^2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} p^0 &= R_2^0 p^2 = R_1^0 R_2^1 p^2 \\ R_2^0 &= R_1^0 R_2^1 \end{aligned}$$

Other basic facts: Orthonomality!

$$R^{-1} = R^T \qquad \det\{R\} = 1$$

Representation of Rotations

Euler Angles: Roll-Pitch-Yaw Representation



$$
\begin{aligned}
R_1^0 &= R_{z,\phi} R_{y,\theta} R_{x,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}.
\end{aligned}
$$

$c_\phi, s_\phi \ldots$ short form for $\sin(\phi), \cos(\phi)$

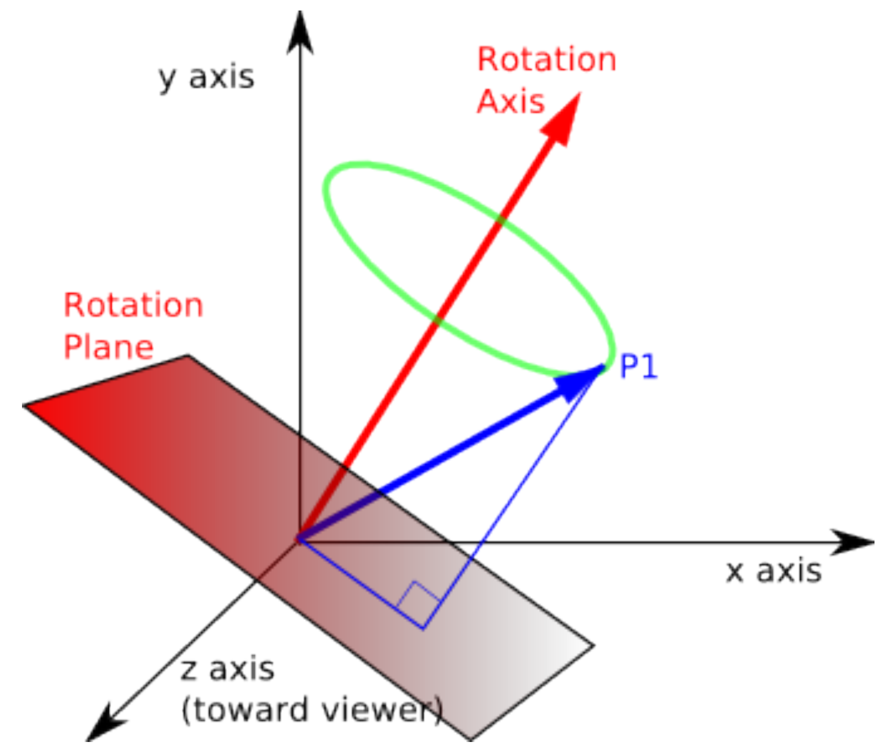
Problems with Euler Angles:

- Not Unique: Many angles result in the same rotation
- Hard to quantify differences between two Euler Angles

Representation of Rotations

Other Types of Representations:

- Angle-Axis
- Unit-Quaternion



Solves the problems of singularities with the Euler Angles

- Easier to compute differences of orientations
- Important if we want to control the orientation of the end-effector

See Siciliano Textbook!

Homogeneous Transformations

- Translations alone are easy $\quad \mathbf{p}^0 = \boldsymbol{\delta}^0 + \mathbf{p}^1$

- Combining Translation and Rotation is a mess...

$$\boldsymbol{p}^0 = \boldsymbol{\delta}^0 + \boldsymbol{R}_1^0(\boldsymbol{\delta}^1 + \boldsymbol{R}_2^1(\boldsymbol{\delta}^2 + \boldsymbol{R}_3^2\boldsymbol{p}^3)))$$

- ...but a trick solves this mess: Homogeneous Transformations!

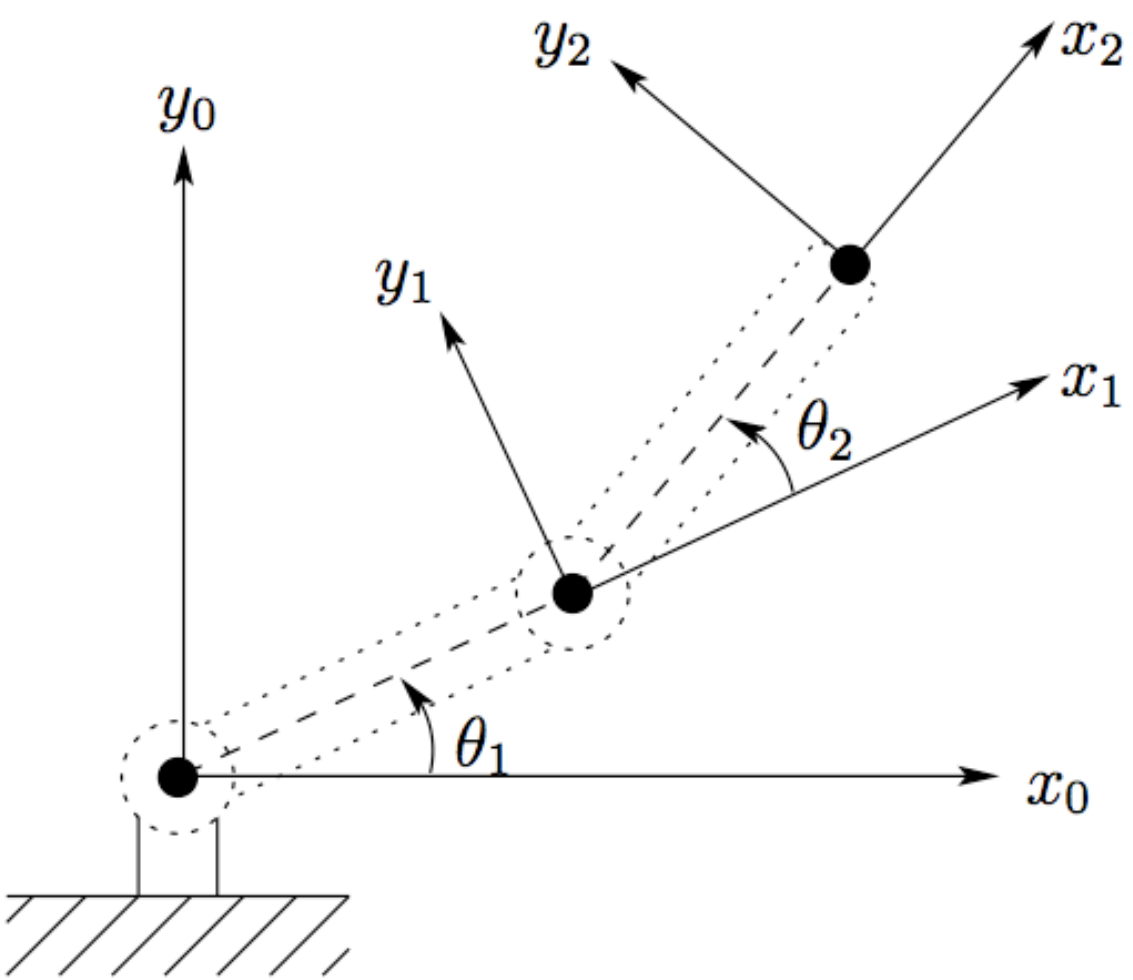
$$\boldsymbol{p}^0 = \boldsymbol{\delta}^0 + \boldsymbol{R}_1^0\boldsymbol{p}^1 \quad \Rightarrow \quad \begin{bmatrix} \boldsymbol{p}^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{\delta}^0 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}^1 \\ 1 \end{bmatrix}$$

$$= \boldsymbol{H}_1^0 \tilde{\boldsymbol{p}}^1$$

4x4 Transformationmatrix

- Hence, we have: $\quad \tilde{\boldsymbol{p}}^0 = \boldsymbol{H}_1^0 \boldsymbol{H}_2^1 \ldots \boldsymbol{H}_n^{n-1} \tilde{\boldsymbol{p}}^n$

Example 2 - *revisited!*



$$\mathbf{A}_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

$$\boldsymbol{H}_1^0 = \boldsymbol{A}_1$$
$$\boldsymbol{H}_2^0 = \boldsymbol{A}_1\boldsymbol{A}_2$$

Differential Forward Kinematics

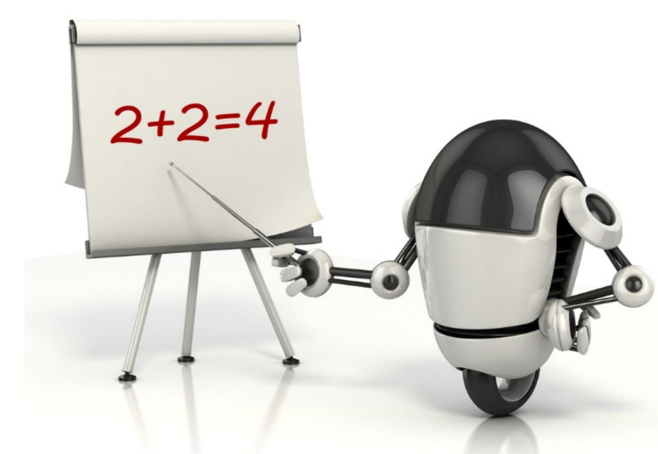
- Sometimes, we are interested in the velocity $\dot{\mathbf{x}}$ or acceleration $\ddot{\mathbf{x}}$

- Remember chain rule from high school?

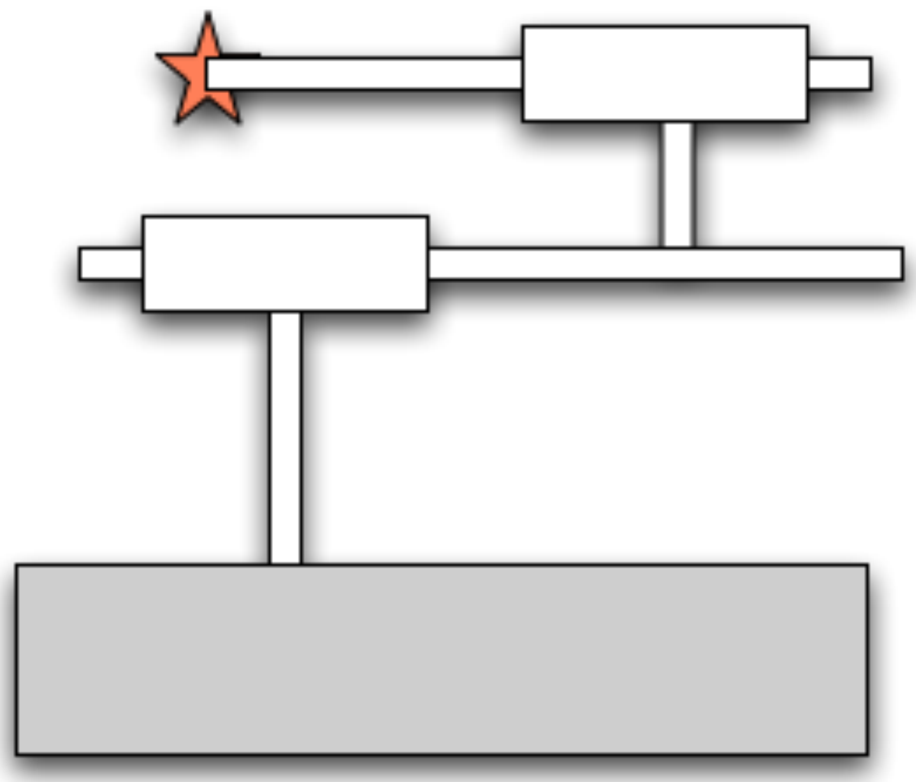
  - **Velocity:** $$\dot{\boldsymbol{x}} = \frac{d}{dt} f(\boldsymbol{q}) = \frac{df(\boldsymbol{q})}{d\boldsymbol{q}} \frac{d\boldsymbol{q}}{dt} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{df(\boldsymbol{q})}{d\boldsymbol{q}} \quad \ldots \ \text{Jacobian}$$

  - **Acceleration:** $$\ddot{\mathbf{x}} = \dot{\mathbf{J}}(\mathbf{q})\dot{\mathbf{q}} + \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}}$$

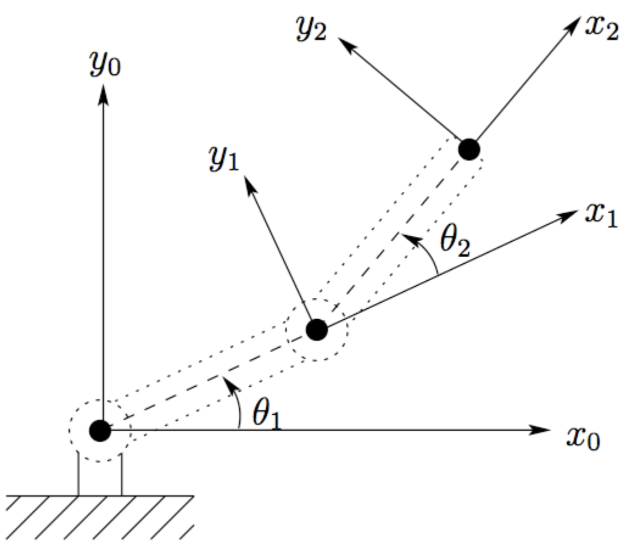


Example 1 - *revisited*



$$\begin{aligned} x &= q_1 + q_2 \\ \dot{x} &= \dot{q}_1 + \dot{q}_2 \\ &= [1, 1] \left[\begin{array}{c} \dot{q}_1 \\ \dot{q}_2 \end{array} \right] = \mathbf{J}\dot{\mathbf{q}} \end{aligned}$$

Examples 2 - *revisited*



$$\boxed{\begin{aligned} x &= x_2 = a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ y &= y_2 = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) \end{aligned}}$$

$$\Downarrow$$

$$\begin{aligned} \dot{x} &= -a_1 \sin\theta_1 \dot{\theta}_1 - a_2 \sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \\ \dot{y} &= a_1 \cos\theta_1 \dot{\theta}_1 + a_2 \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \end{aligned}$$

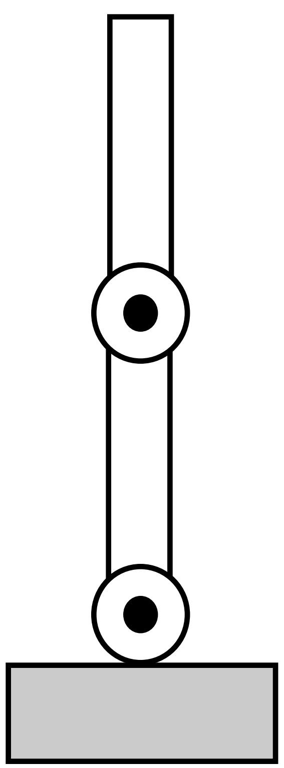
$$\Downarrow$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -a_1 \sin(\theta_1) - a_2 \sin(\theta_1 + \theta_2) & -a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) & +a_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Singularities

- What happens when I stretch out my arm?

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -(a_1 + a_2)\sin(\theta_1) & -a_2 \sin(\theta_1) \\ (a_1 + a_2)\cos(\theta_1) & +a_2 \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$



- The colums of the Jacobian get linearly dependent
- I loose a degree of freedom and

$$\det \mathbf{J} = 0$$

- These positions are called ***Singularities***!

Computing the Jacobians

Two ways are common:

- **Analytical Jacobians** are easier to understand (as before) and can be derived by symbolic differentiation. However, the representation of the rotation matrix can cause “representational singularities”

- **Geometric Jacobians** are derived from geometric insight (more contrived), can be implemented easier and do not have “representational singularities”.

- **Main difference:** How the Jacobian for the orientation is represented

See the Siciliano Textbook...

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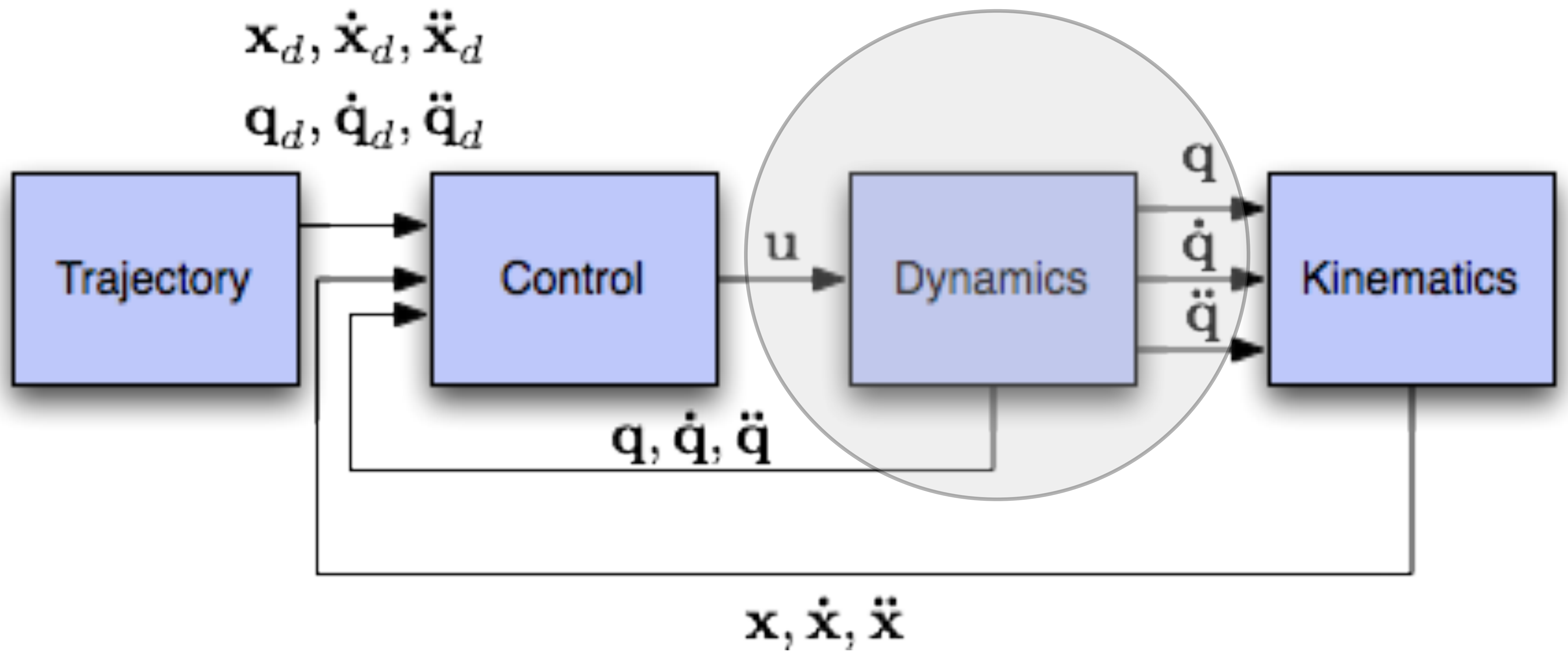
Block Diagram of Complete System



Dynamics

- Forward dynamics model

$$\ddot{\boldsymbol{q}} = f(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{u})$$

- Essential equations:
  - Forces F_i (Kraft):

mass

$$m\ddot{x} = \sum_i F_i$$

  - Torques τ_i (Drehmoment):

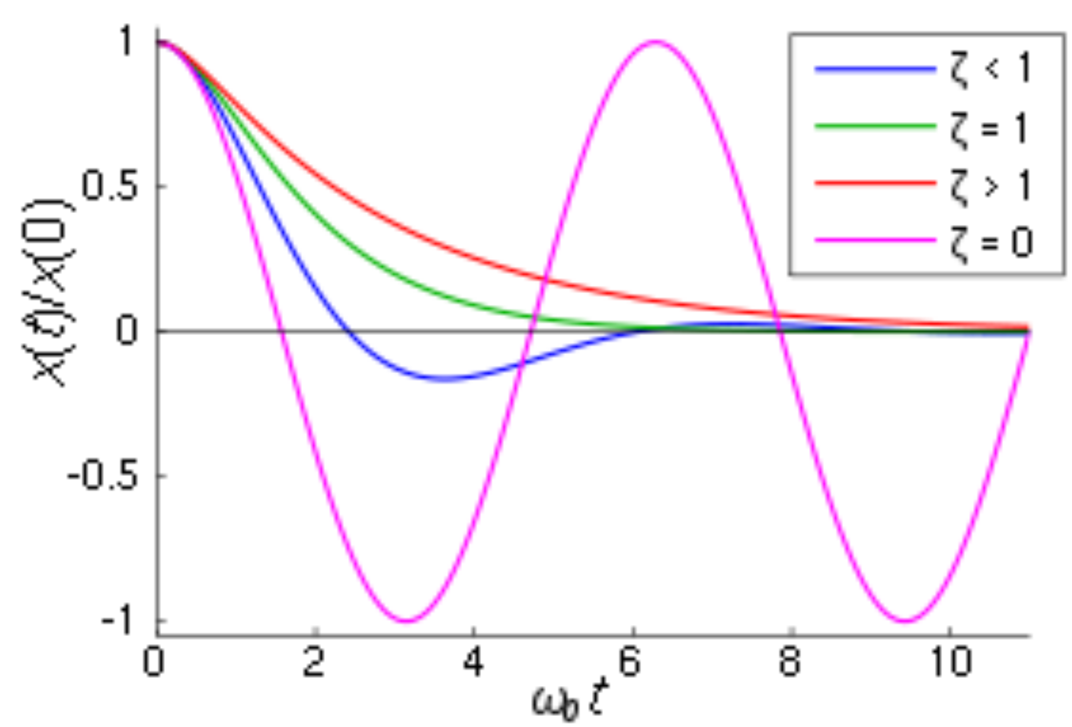
Inertia

$$I\ddot{q} = \sum_i \tau_i$$

What forces are there?

- **Gravity:** $F_{\text{grav}} = mg$
- **Friction**
  - Stiction: $F_{\text{stiction}} = -c_s \text{sgn}(\dot{x})$
  - Damping (Viscous Friction): $F_{\text{damping}} = -D\dot{x}$
- **Springs:**
- **Example:** Spring-Damper System

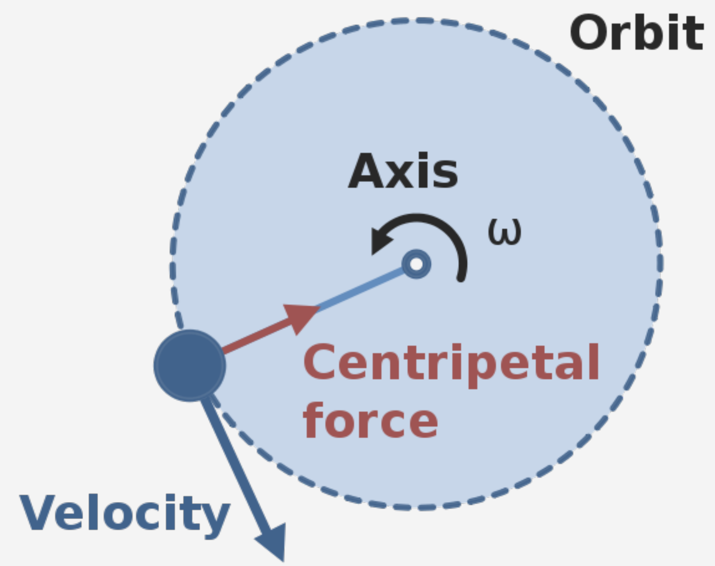
$$m\ddot{x} = K(x_{\text{eq}} - x) - D\dot{x}$$



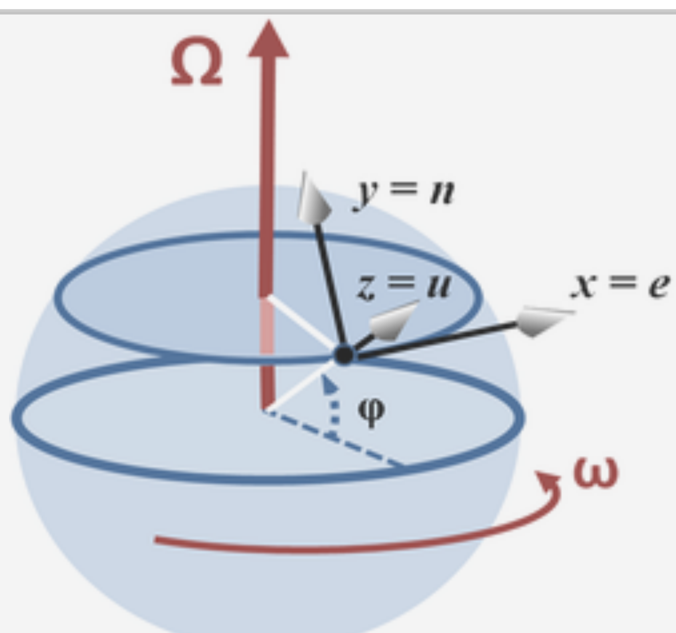
What torques are there?

- Gravity $\tau_{\text{gravity}} = mgl$
- Friction just as before.
- **Virtual Forces:**
  - Centripetal
  - Coriolis forces

Centripetal Forces



Coriolis Forces



General Form

Dynamics are usually denoted in this form:

$$\mathbf{u} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q})$$

- Motor commands: $\boldsymbol{u}$
- Joint positions, velocities and accelerations: $\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}$
- Mass matrix: $\boldsymbol{M}(\boldsymbol{q})$
- Coriolis forces and Centrifugal forces: $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$
- Gravity: $\boldsymbol{g}(\boldsymbol{q})$

General Form

- Dynamics are usually denoted in this form:

$$\mathbf{u} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q})$$

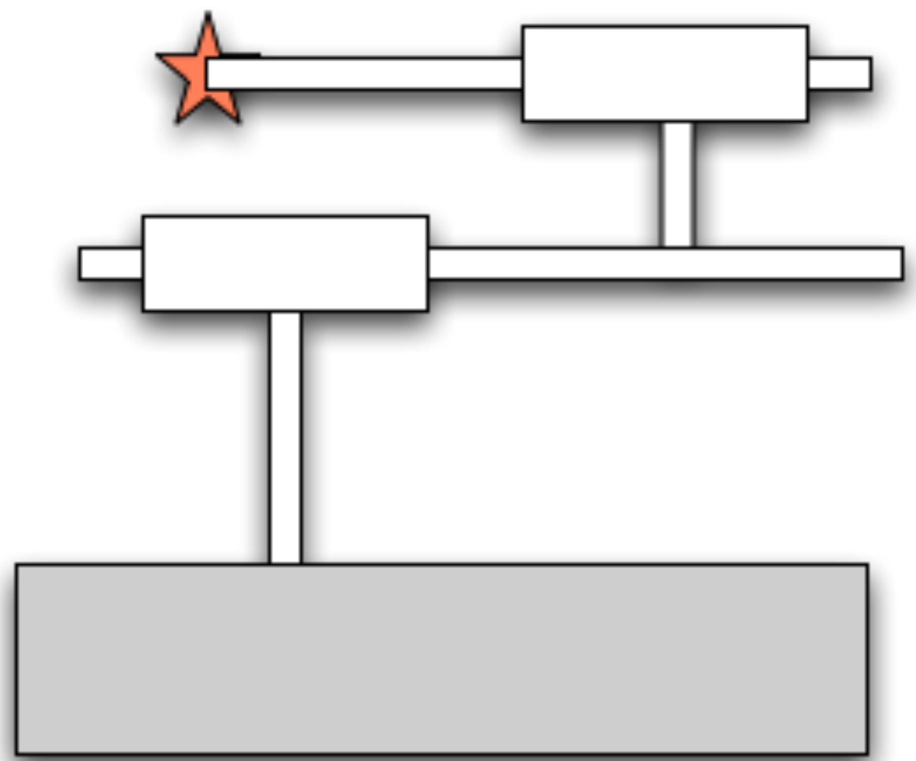
  - Inverse dynamics model $\quad \boldsymbol{u} = f(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$

- From this equation we can already build a robot simulator
  - Forward dynamics model $\quad \ddot{\boldsymbol{q}} = f(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{u})$

Compute accelerations $\quad \ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{q})(\mathbf{u} - \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{g}(\mathbf{q}))$

Integrate $\quad \dot{\mathbf{q}} = \int_0^t \ddot{\mathbf{q}} d\tau, \qquad \mathbf{q} = \int_0^t \dot{\mathbf{q}} d\tau$

Example 1 - *revisited*



Acting Force

$$\begin{aligned} m_1\ddot{x}_1 &= u_1 - u_2 \\ m_2\ddot{x}_2 &= u_2 \end{aligned}$$

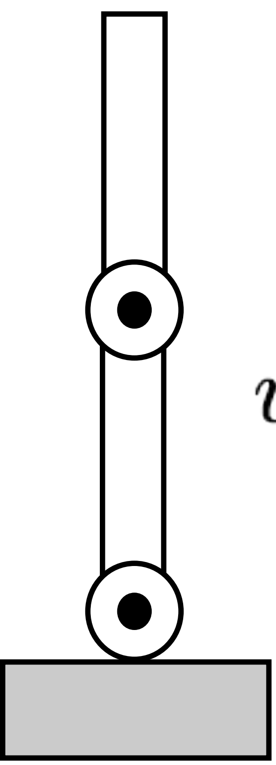
Joints Position

$$\begin{aligned} x_1 &= q_1 \\ x_2 &= q_1 + q_2 \end{aligned}$$

Dynamics

$$\begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Example 2 - *revisited*



$$
\begin{aligned}
u_1 \quad = \quad & [m_1 l_{g1}^2 + J_1 + m_2(l_1^2 + l_{g2}^2 + 2 l_1 l_{g2} \cos\theta_2) + J_2]\ddot{\theta}_1 \\
+ \quad & [m_2(l_{g2}^2 + l_1 l_2 \cos\theta_2) + J_2]\ddot{\theta}_2
\end{aligned}
$$

Inertial Forces

$$- \quad 2 m_2 l_1 l_{g2} \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2$$

Coriolis Forces

$$- \quad 2 m_2 l_1 l_{g2} \dot{\theta}_1^2 \sin\theta_2$$

Centripetal Forces

$$+ \quad m_1 g l_{g1} \cos\theta_1 + m_2 g(l_1 \cos\theta_1 + l_{g2} \cos(\theta_1 + \theta_2))$$

Gravity

$$
\begin{aligned}
u_2 \quad = \quad & [m_2(l_{g2}^2 + l_1 l_{g2} \cos\theta_2) + J_2]\ddot{\theta}_1 \\
+ \quad & (m_2 l_{g2}^2 + J_2)\ddot{\theta}_2
\end{aligned}
$$

Inertial Forces

$$- \quad m_2 l_1 l_{g2} \dot{\theta}_1^2 \sin\theta_2$$

Centripetal Forces

$$+ \quad m_2 g l_{g2} \cos(\theta_1 + \theta_2)$$

Gravity

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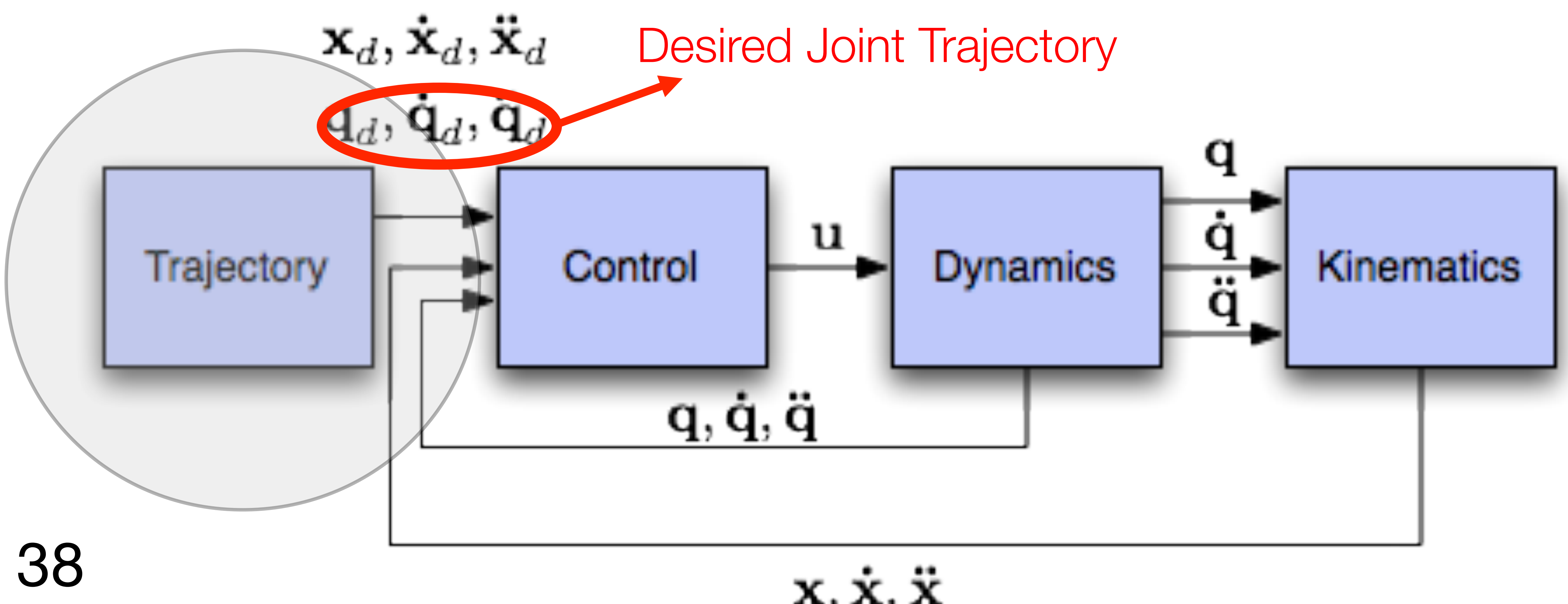
Block Diagram of Complete System

Trajectory: $\boldsymbol{\tau} = \boldsymbol{q}_{1:T}$

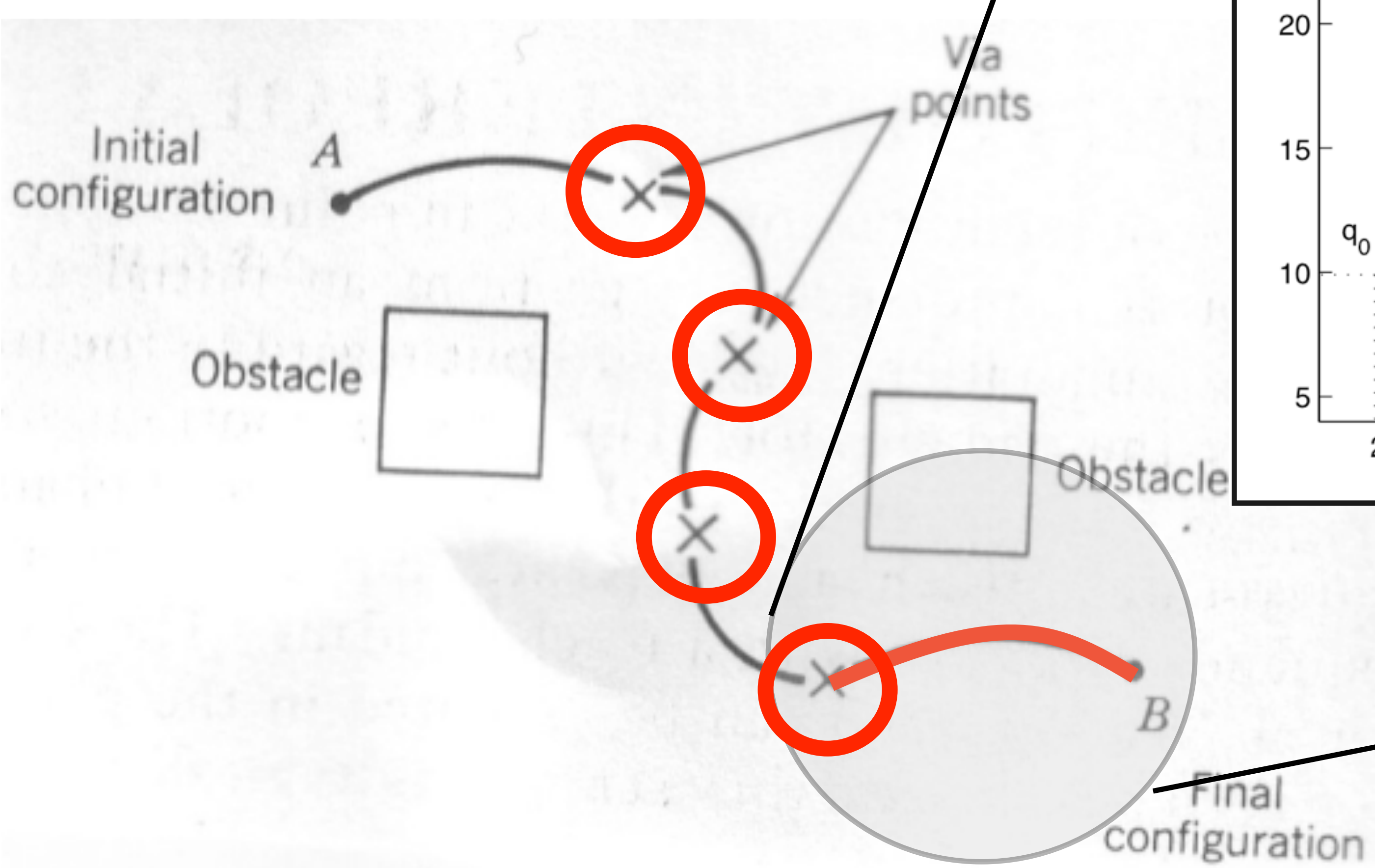
- Specifies the joint positions for each time step t
- Used to specify the desired movement plan
- Inherently includes velocities and accelerations

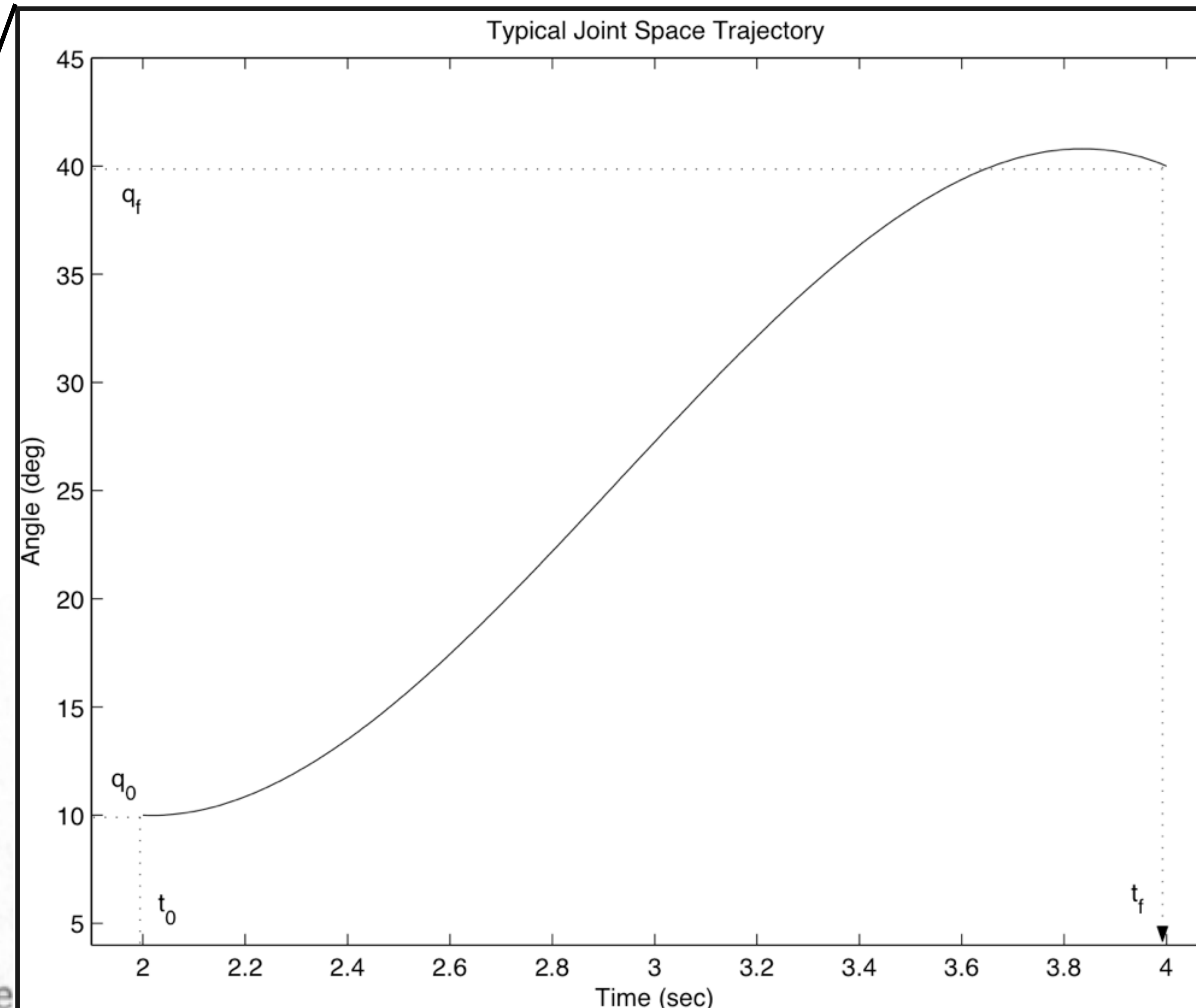


Movement Plans

How to represent trajectories ?

➡ Representation with via-points





Trajectory of a single segment

What do we need?

Look once again at the mathematical model of a robot:

$$\begin{aligned} \ddot{\mathbf{q}} &= \mathbf{M}^{-1}(\mathbf{q})\mathbf{u} \\ \dot{\mathbf{q}} &= \int_0^t \ddot{\mathbf{q}} d\tau, \qquad \mathbf{q} = \int_0^t \dot{\mathbf{q}} d\tau \end{aligned}$$

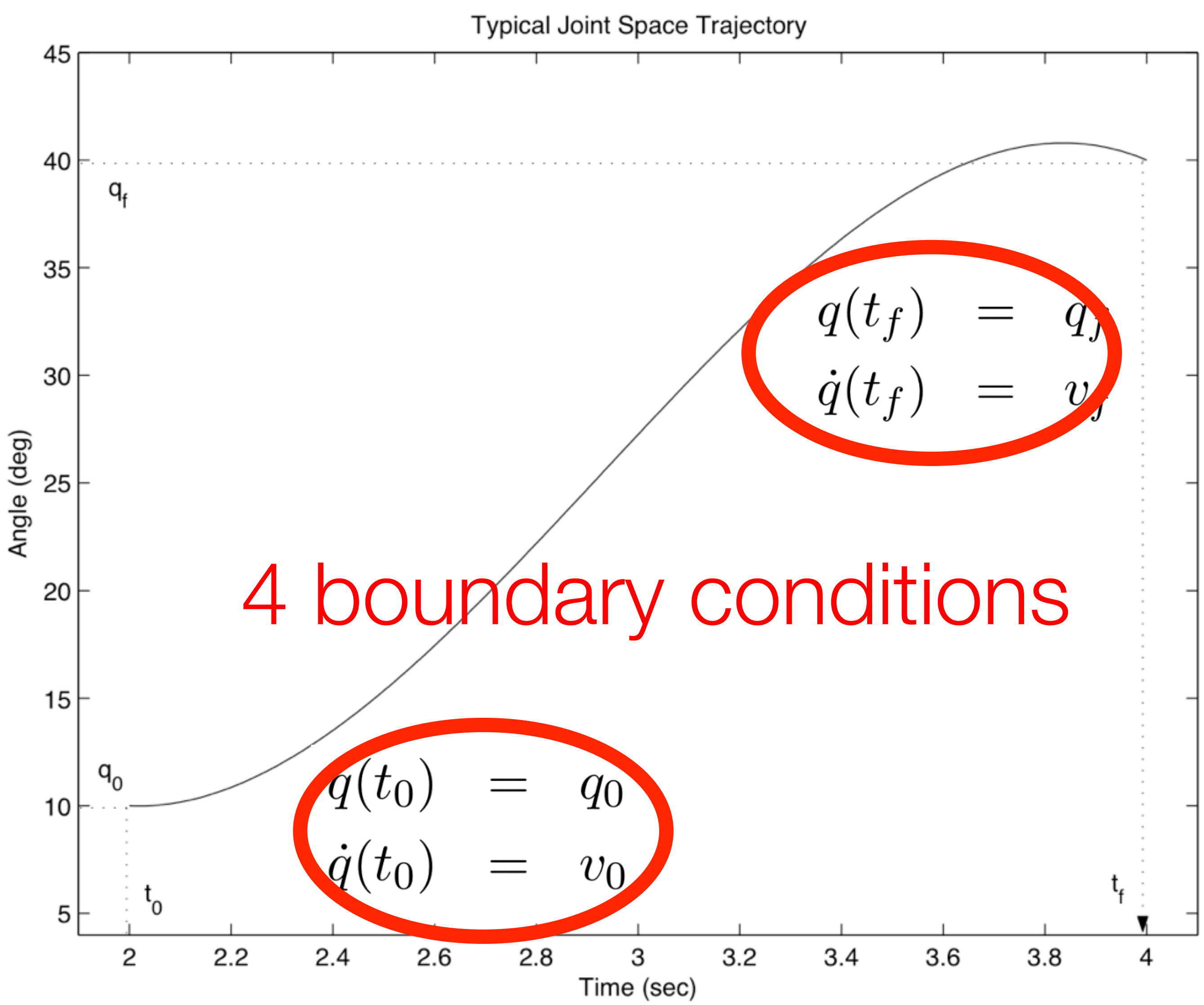
- ➡ Our motor commands can only **influence the acceleration!**
- ➡ The velocities and positions are just integrals of the acceleration.

➡ Any trajectory representation must be **twice differentiable!** The positions and velocities cannot jump.

➡ We can use **polynomials**!

Cubic Splines

How do guarantee no jumps in pos. and vel.?



$$\begin{aligned} q(t_f) &= q_f \\ \dot{q}(t_f) &= v_f \end{aligned}$$

4 boundary conditions

$$\begin{aligned} q(t_0) &= q_0 \\ \dot{q}(t_0) &= v_0 \end{aligned}$$

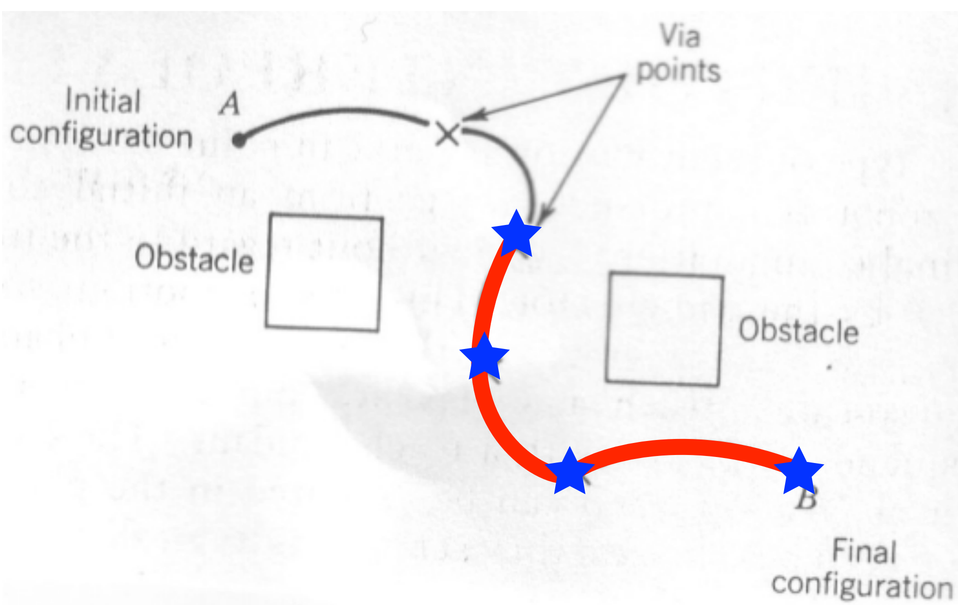
4 free parameters

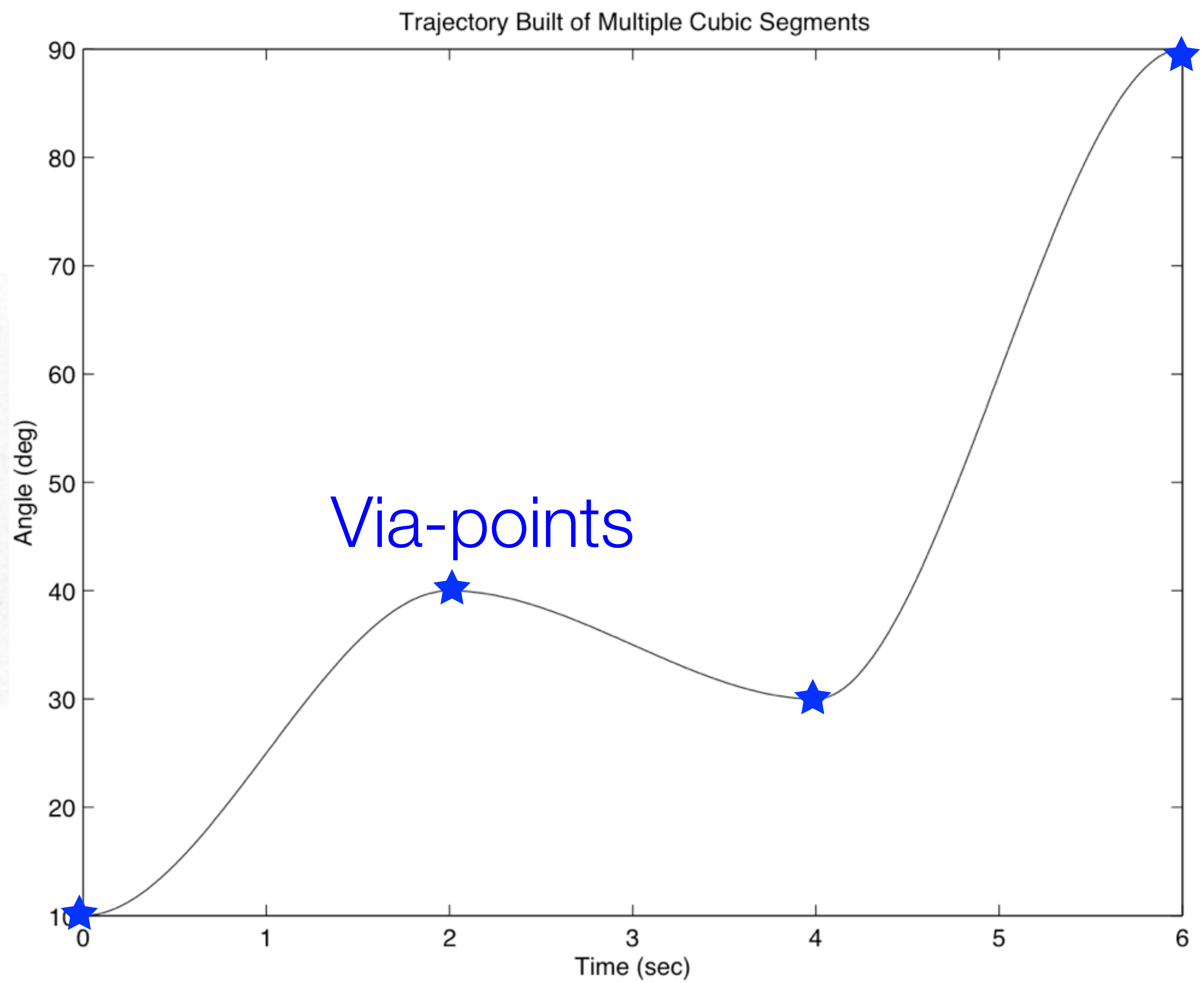
$$\begin{aligned} q(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\ \dot{q}(t) &= a_1 + 2a_2 t + 3a_3 t^2 \end{aligned}$$

Solve using Boundary Conditions

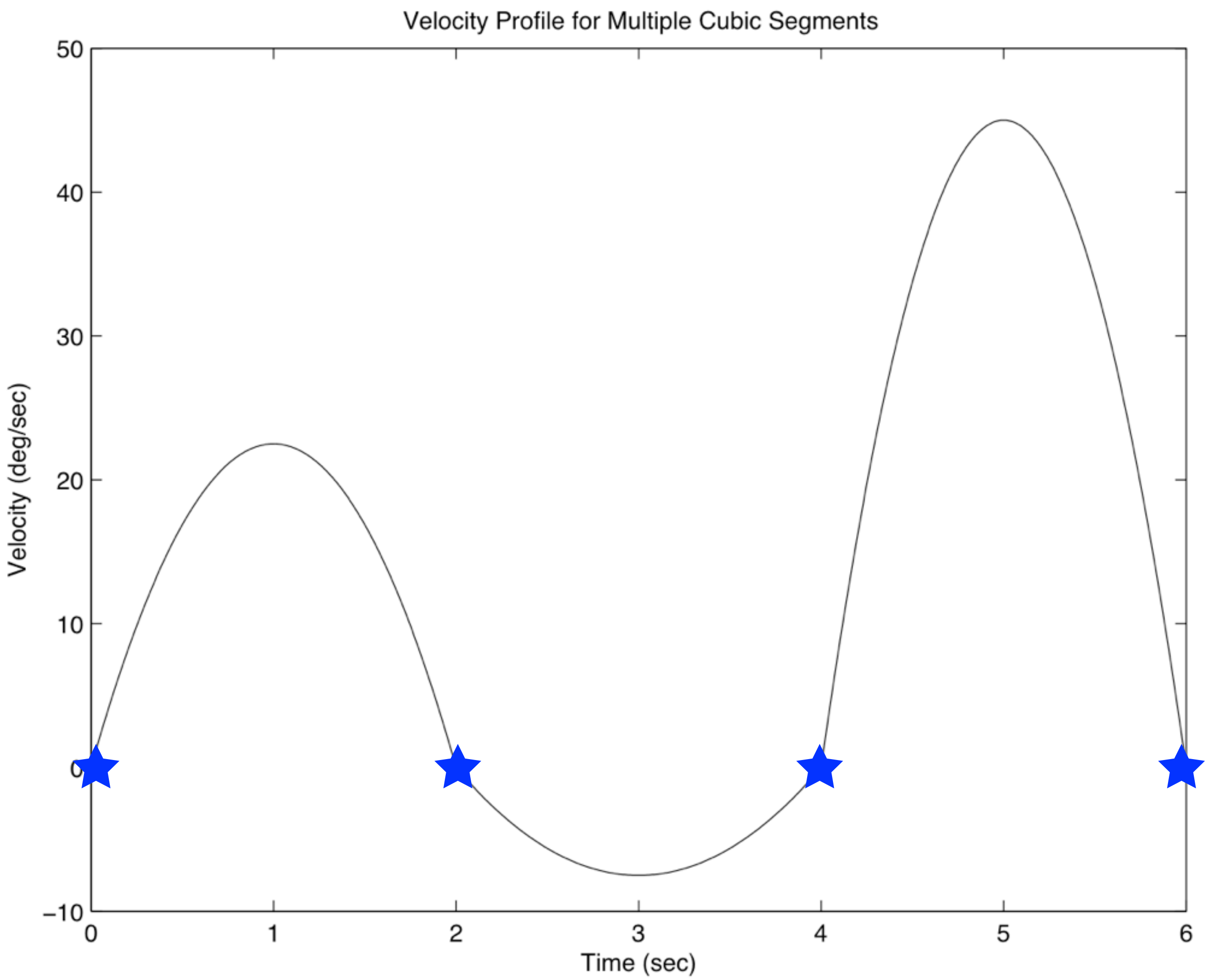
$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ v_0 \\ q_f \\ v_f \end{bmatrix}$$

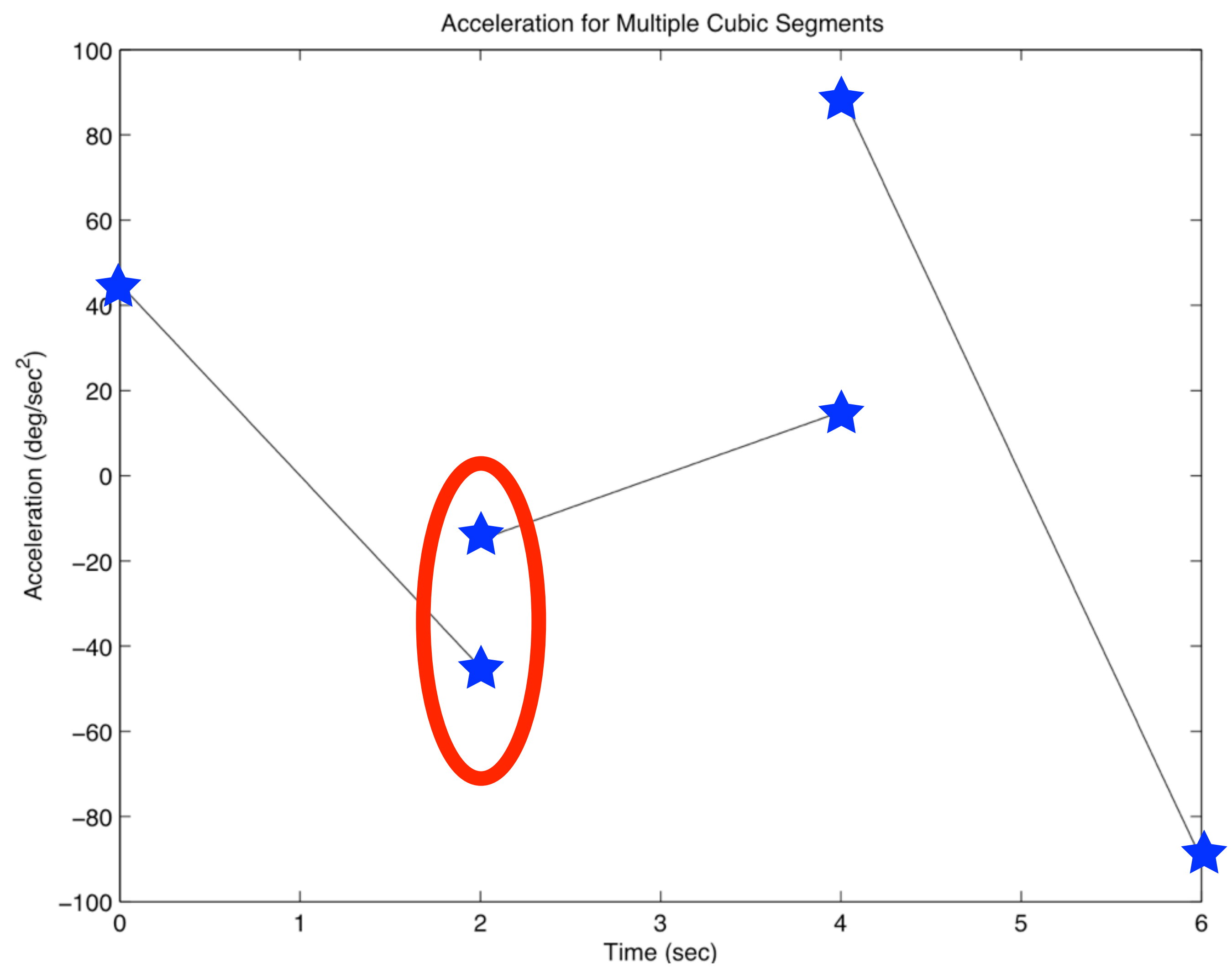
Problems with Cubic Splines





Problems with Cubic Splines



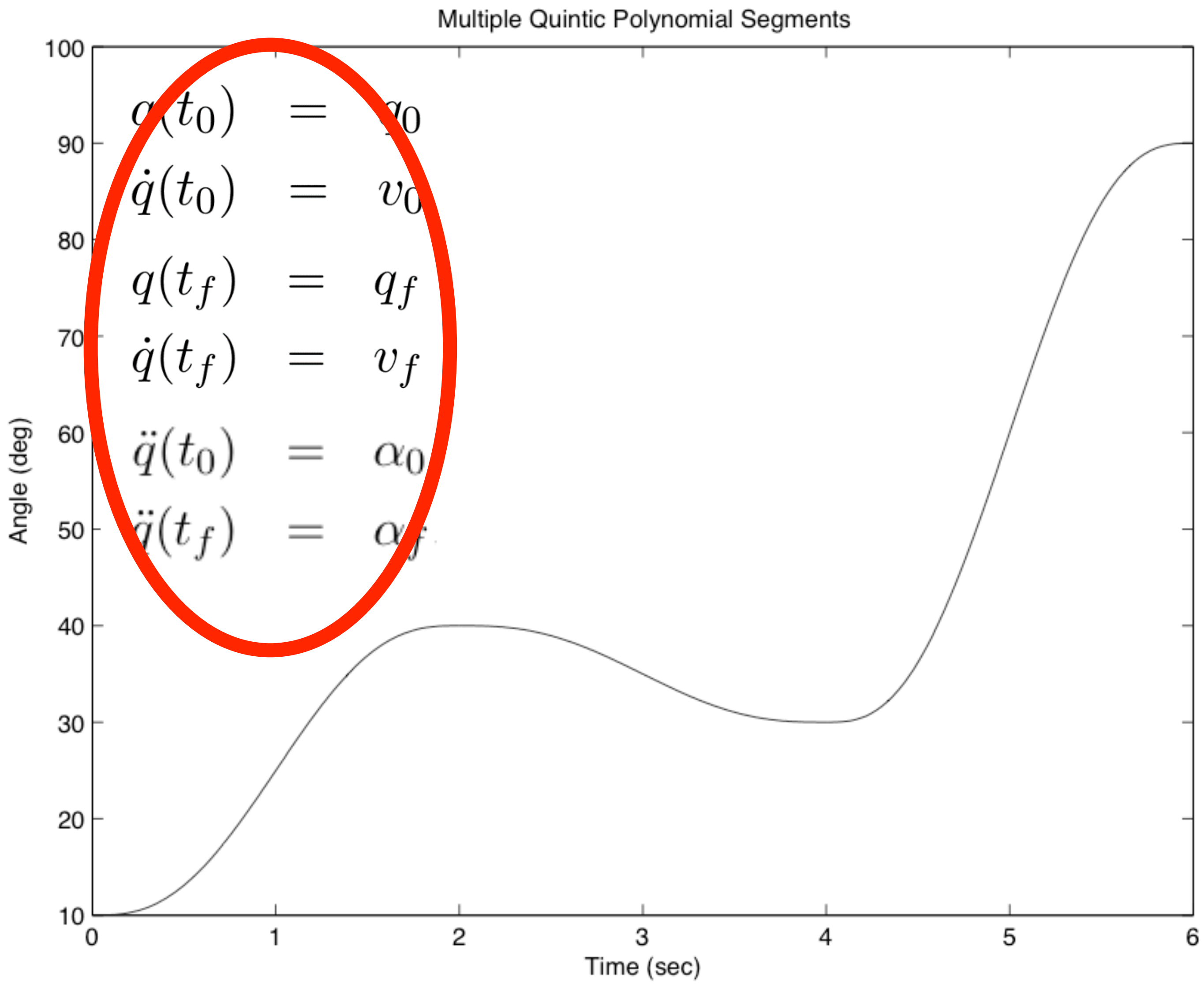


We still get jumps in the acceleration!

- ➡ Dangerous at high speed and damage the robot
- ➡ This requires higher order splines...

Quintic Splines

No jumps in the acceleration ➡ 6 boundary conditions



$$
\begin{aligned}
q(t_0) &= q_0 \\
\dot{q}(t_0) &= v_0 \\
q(t_f) &= q_f \\
\dot{q}(t_f) &= v_f \\
\ddot{q}(t_0) &= \alpha_0 \\
\ddot{q}(t_f) &= \alpha_f
\end{aligned}
$$

Replace Cubic Polynomials by Quintic Polynomials

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

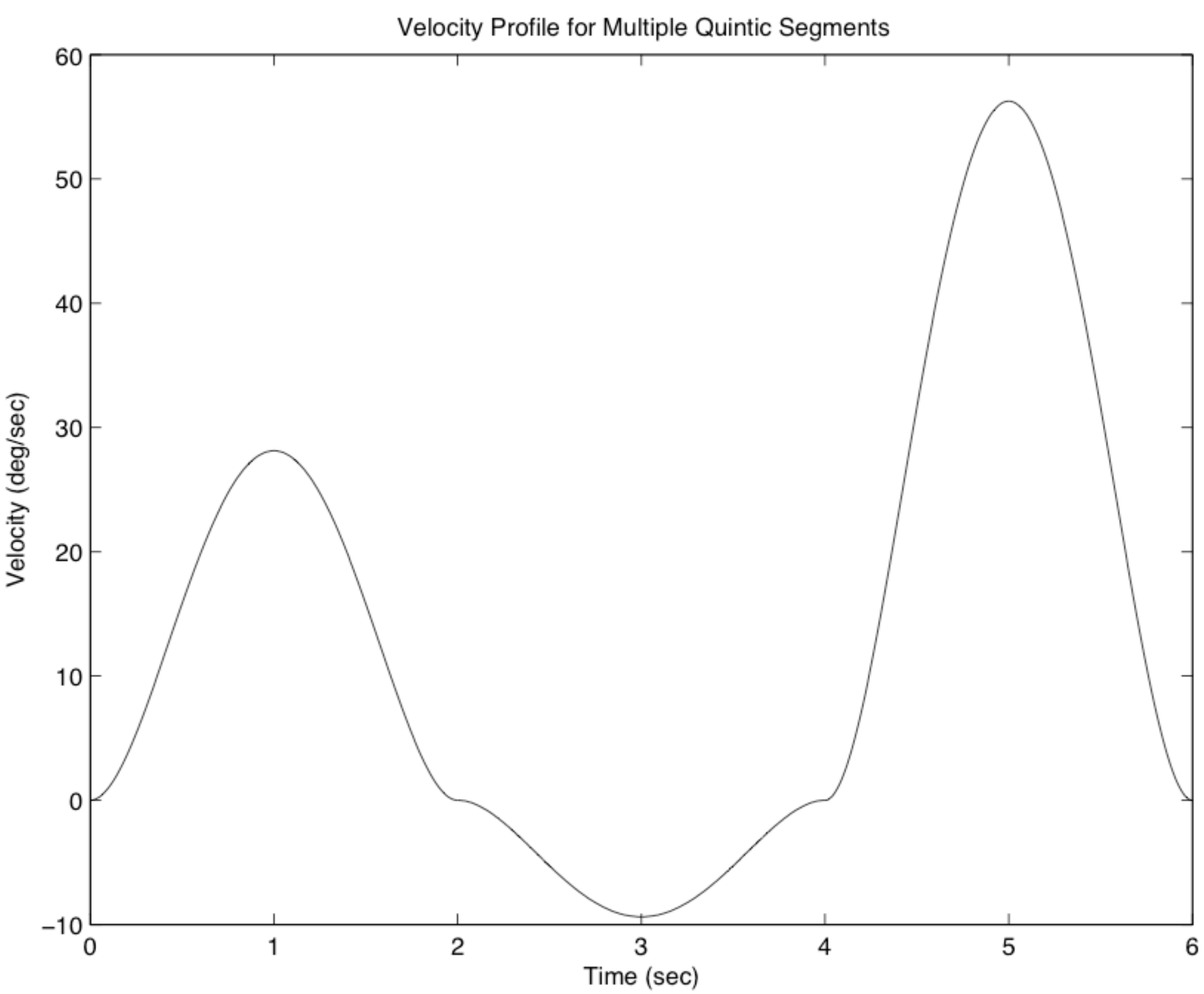
6 free parameters

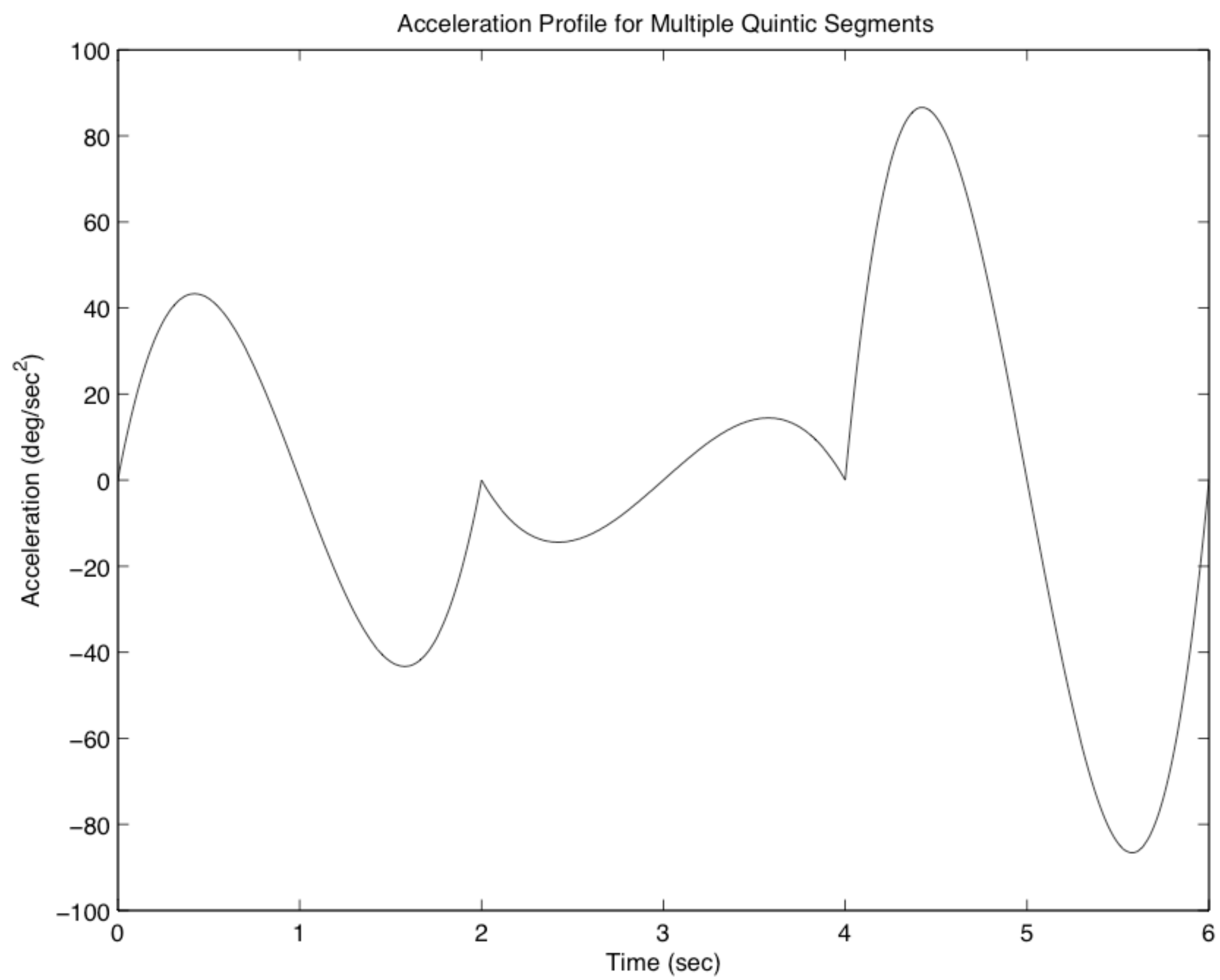
Use new boundary conditions

$$
\begin{bmatrix}
1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t_0^5 \\
0 & 1 & 2t_0 & 3t_0^2 & 4t_0^3 & 5t_0^4 \\
0 & 0 & 2 & 6t_0 & 12t_0^2 & 20t_0^3 \\
1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\
0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 & 5t_f^4 \\
0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}
=
\begin{bmatrix} q_0 \\ v_0 \\ \alpha_0 \\ q_f \\ v_f \\ \alpha_f \end{bmatrix}
$$

Quintic Splines

Smooth velocity and acceleration profiles with quintic splines





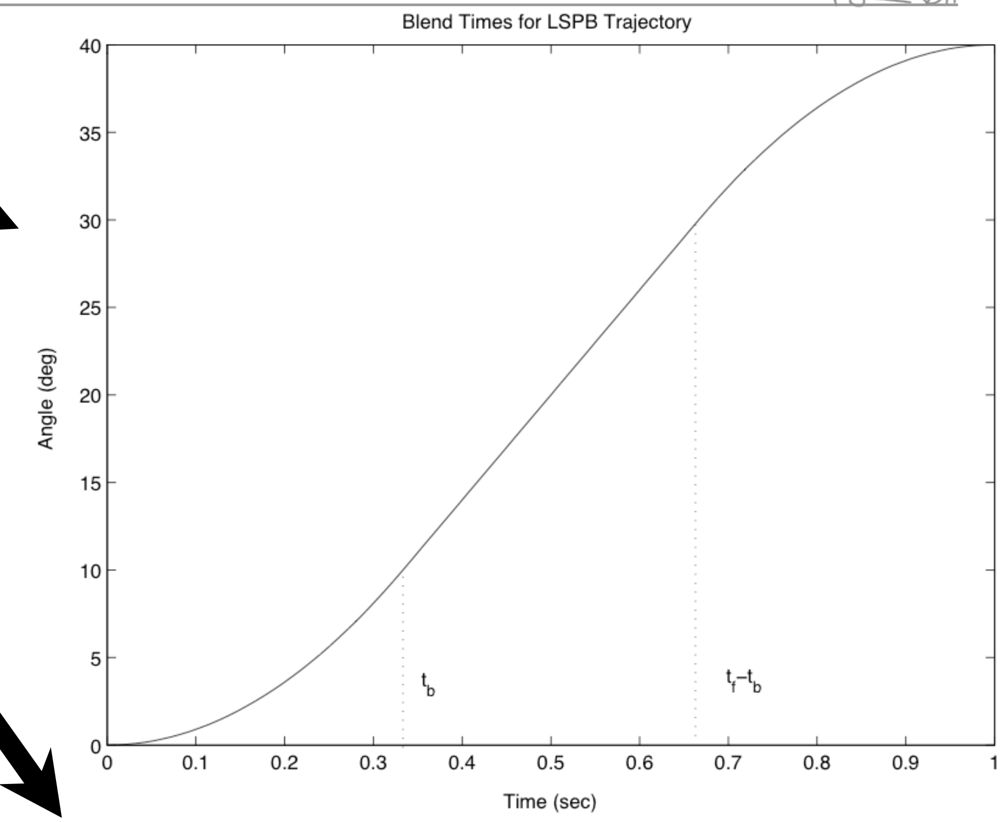
Alternatives to Splines

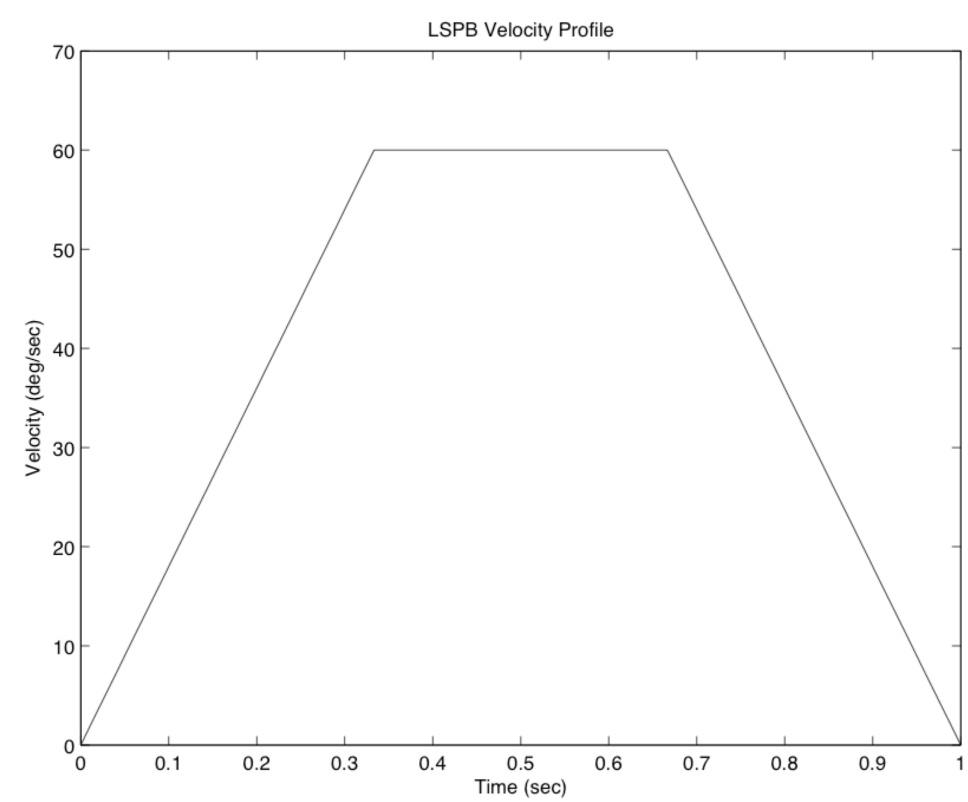
- Linear Segments with Parabolic Blends!
- Trapezoidal Minimum Time Trajectories
- Potential Fields $V(\mathbf{q})$

$$\dot{\mathbf{q}} = \frac{dV(\mathbf{q})}{d\mathbf{q}}$$

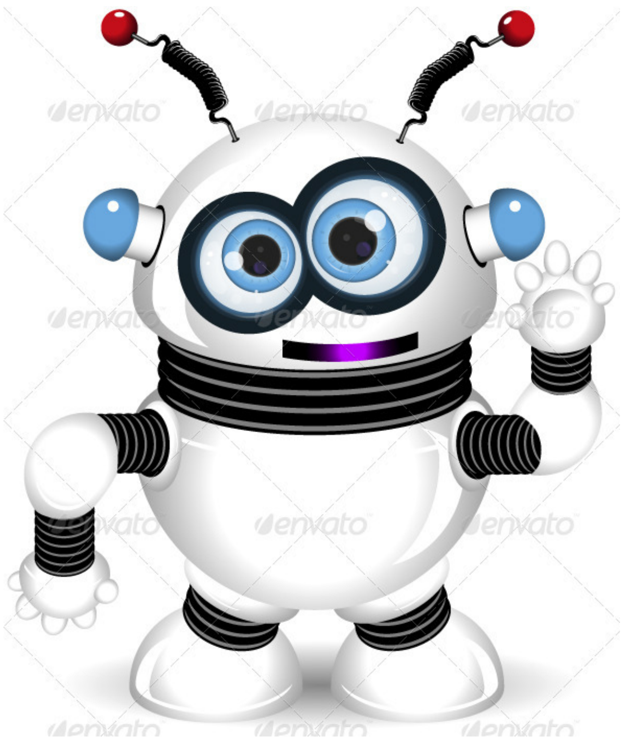
- Nonlinear Dynamical Systems

$$\ddot{\mathbf{q}} = f(\mathbf{q}, \dot{\mathbf{q}}, \theta)$$





Ask questions...



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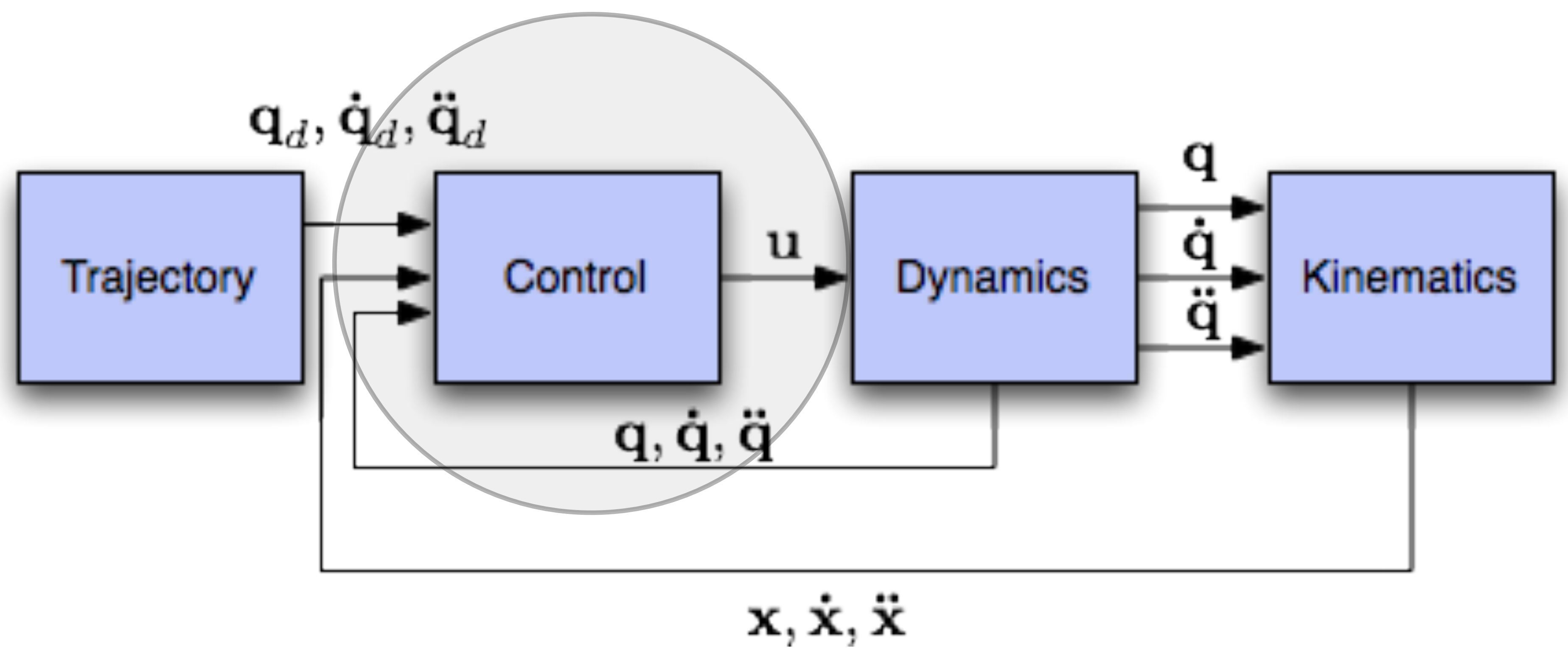
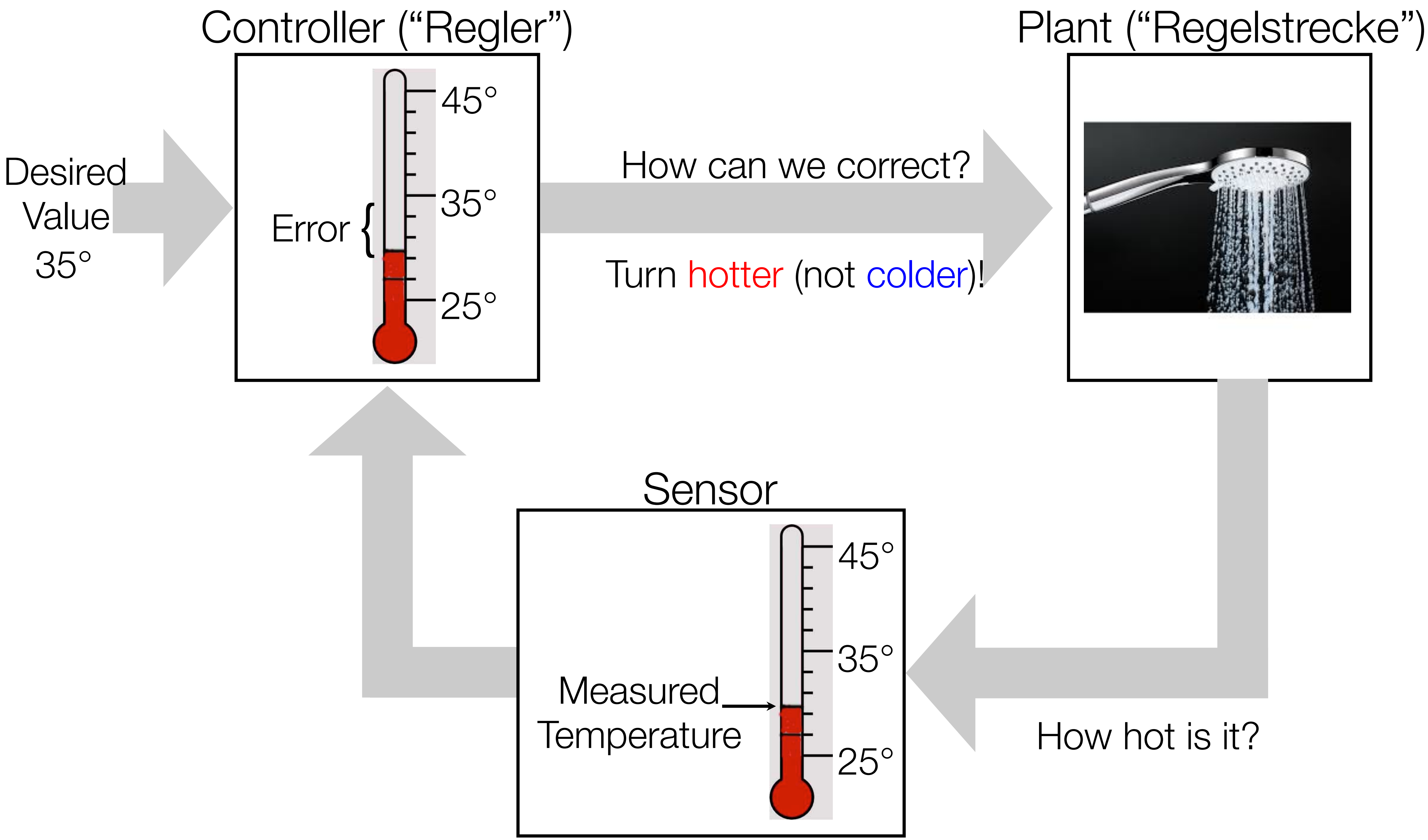
Control

Why do we need control?

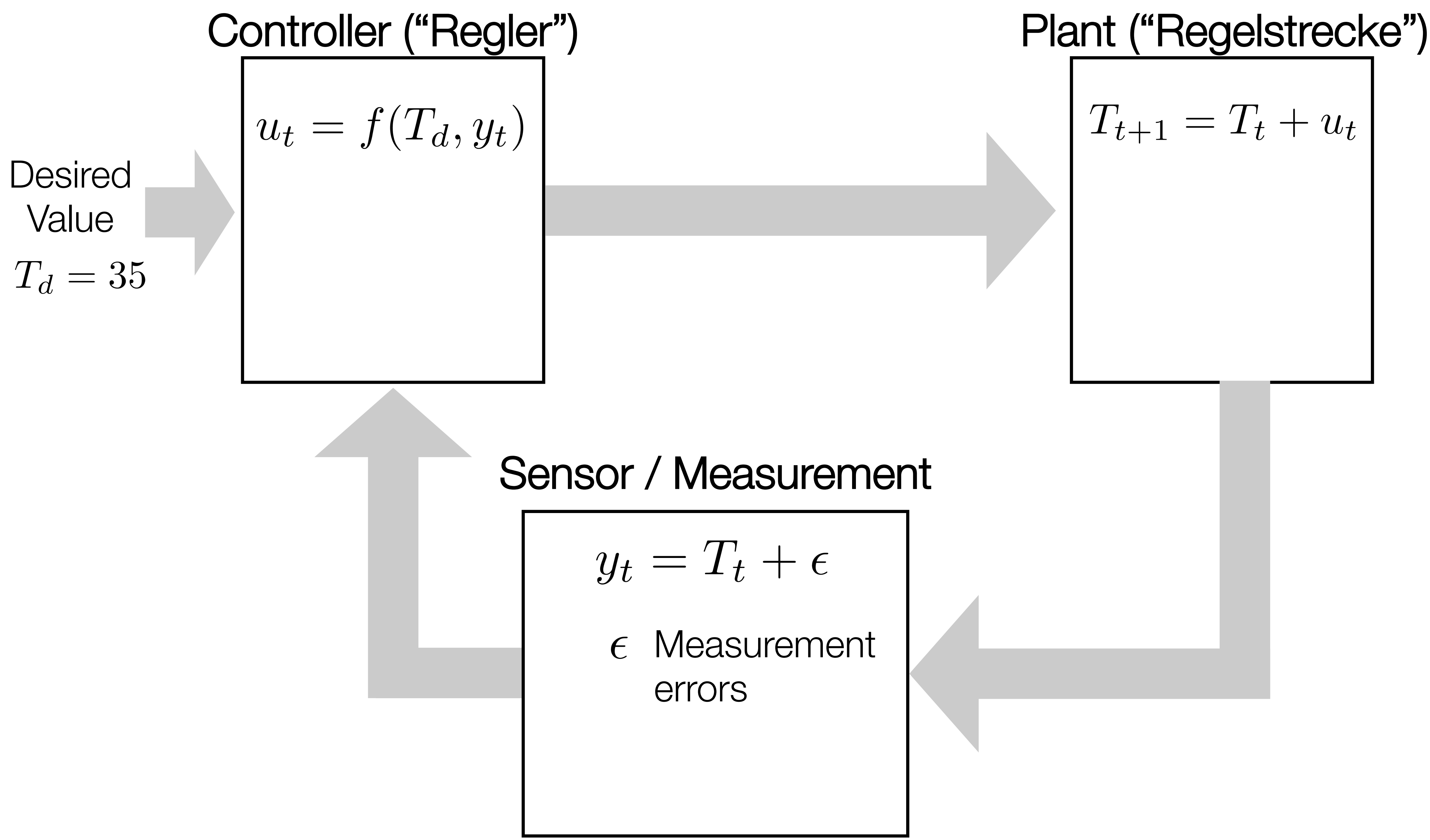
➡ Given a desired trajectory $\boldsymbol{\tau}_d$ we still need to find the controls $\boldsymbol{u}$ to follow this trajectory



Feedback Control: Generic Idea



Feedback Control: Generic Idea



Linear Feedback Control

Controller ("Regler")

$u_t = K(T_d - y_t)$

Plant ("Regelstrecke")

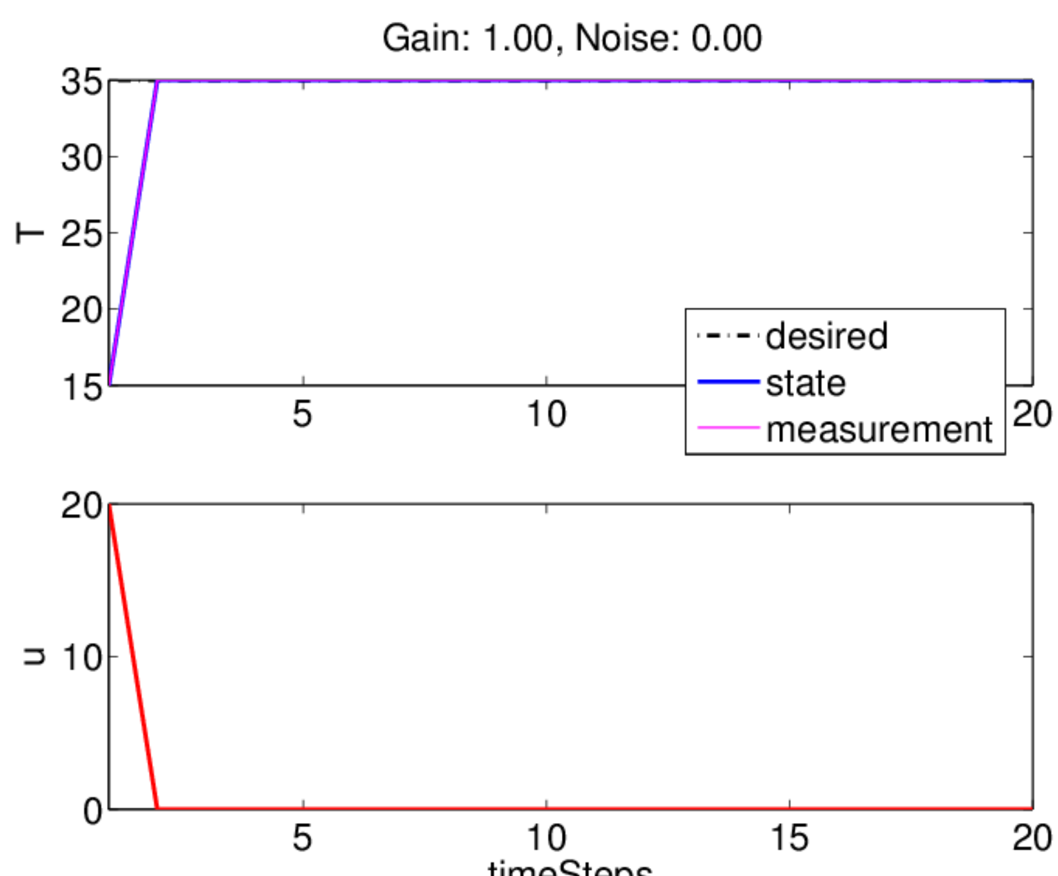
$T_{t+1} = T_t + u_t$

Desired Value

$T_d = 35$

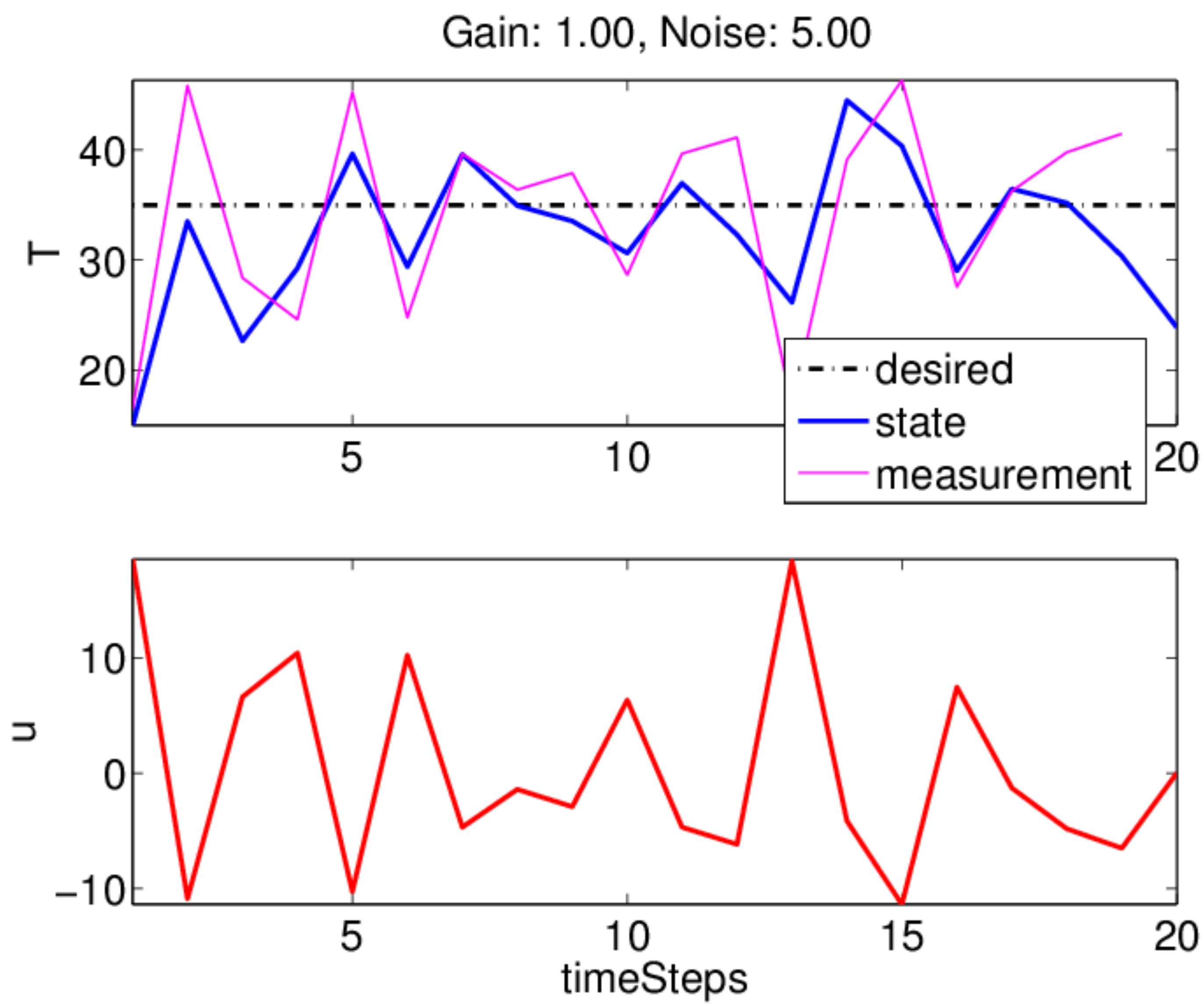
Sensor / Measurement

$y_t = T_t + \epsilon$



Measurement Errors

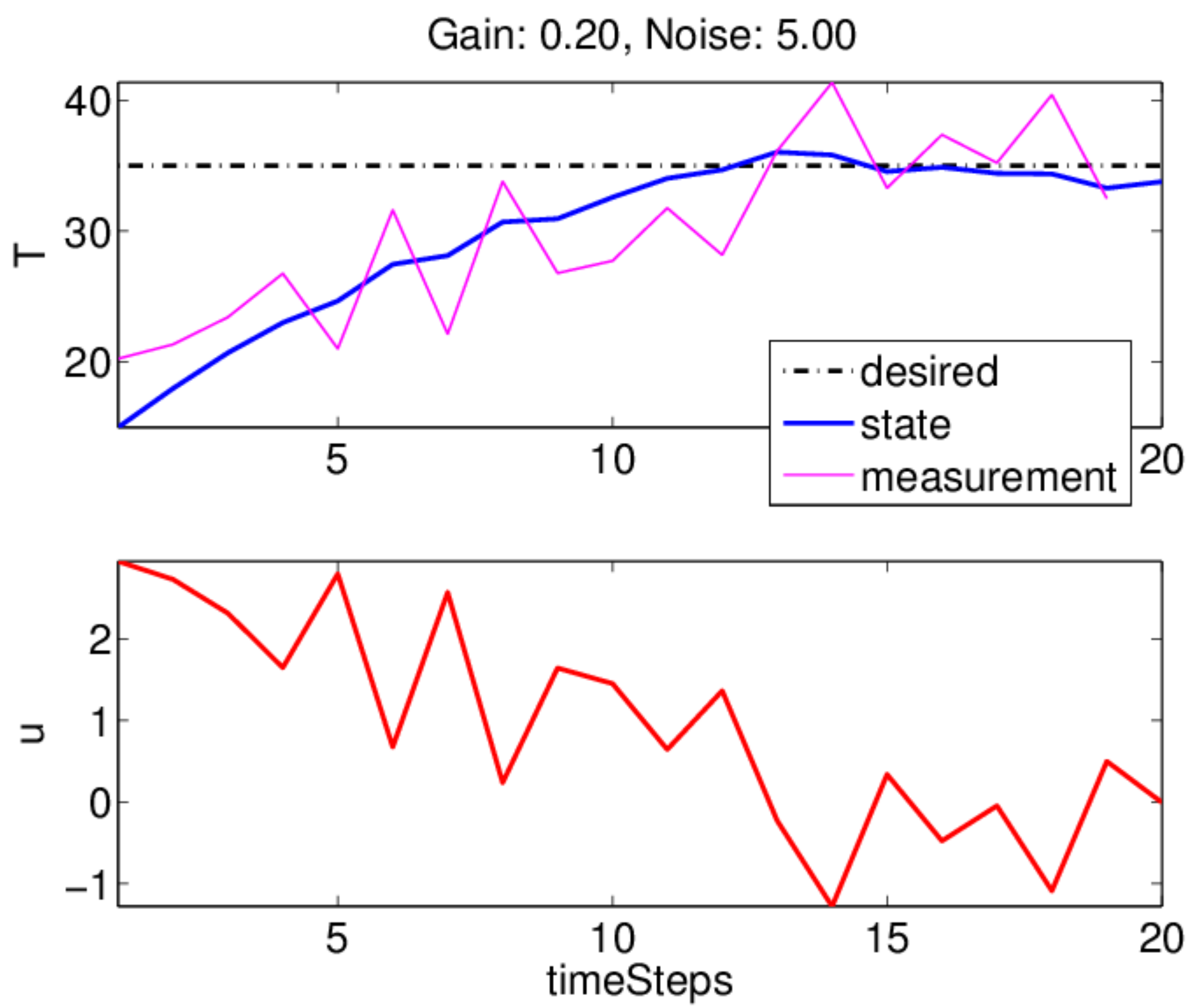
What effect do measurement errors have?



➡ High Motor Commands, that's not a comfortable way to shower

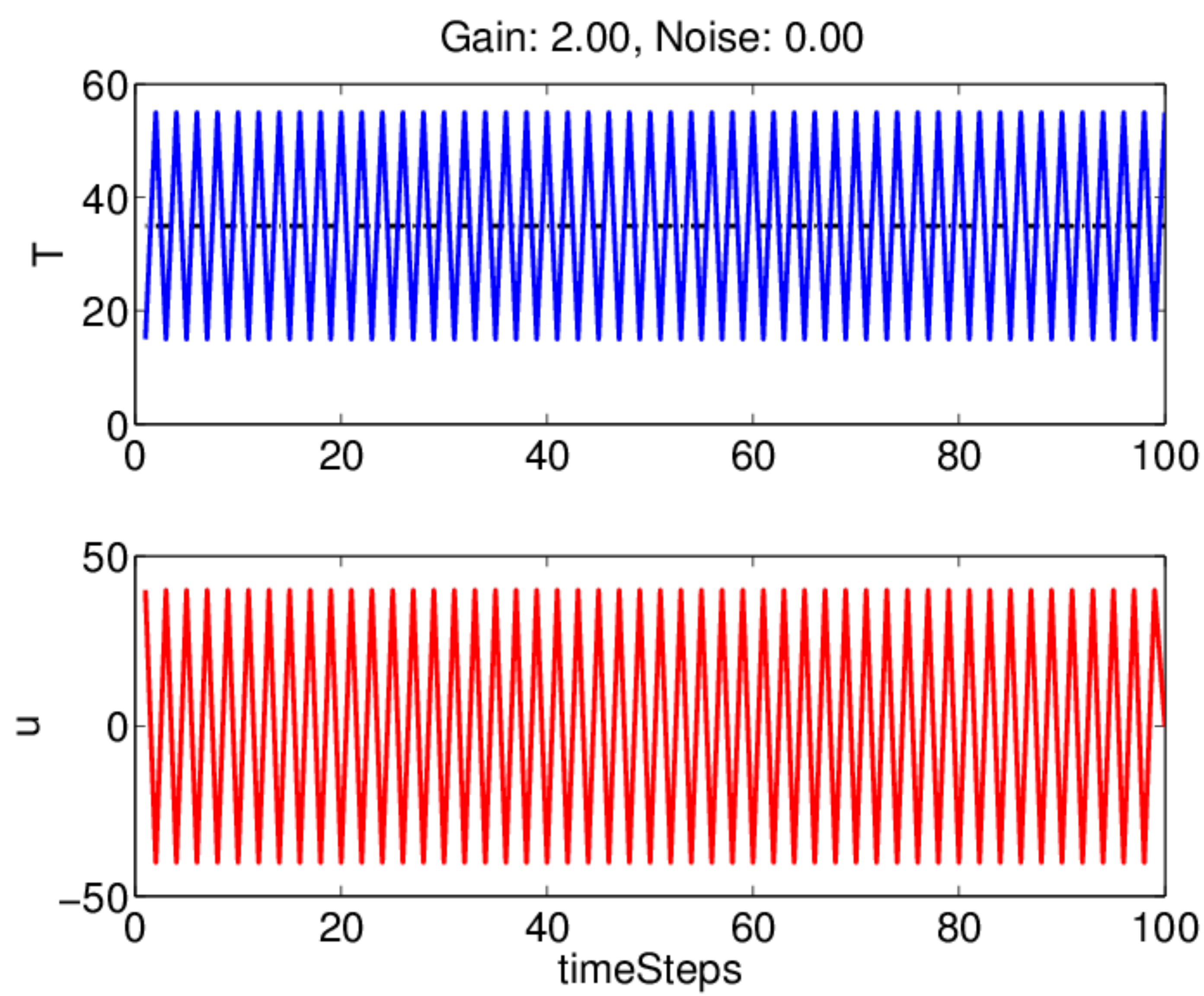
Proper Control with Measurement Errors

Lower our gains!!!



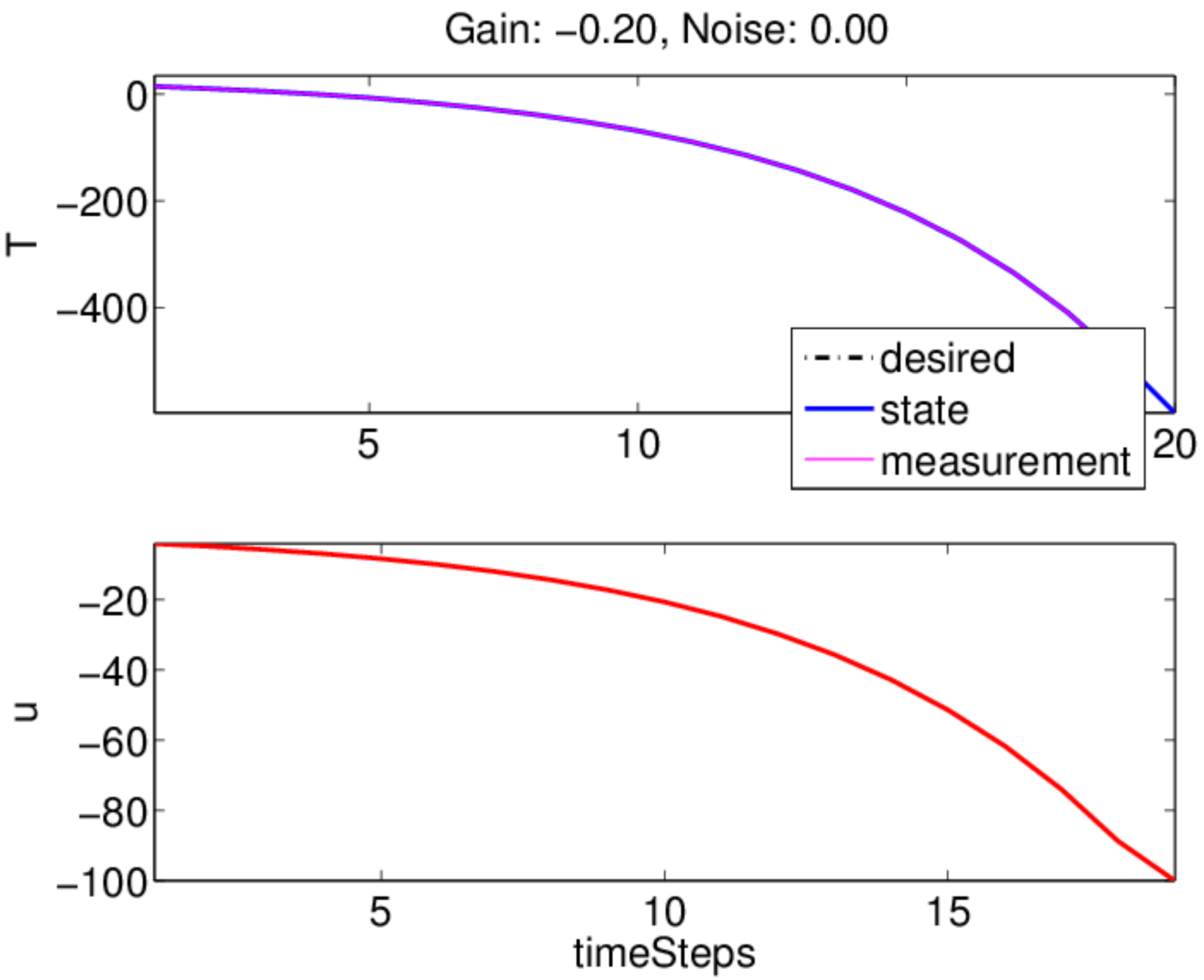
What do High Gains do?

High gains are always problematic!!!! Check K = 2 !

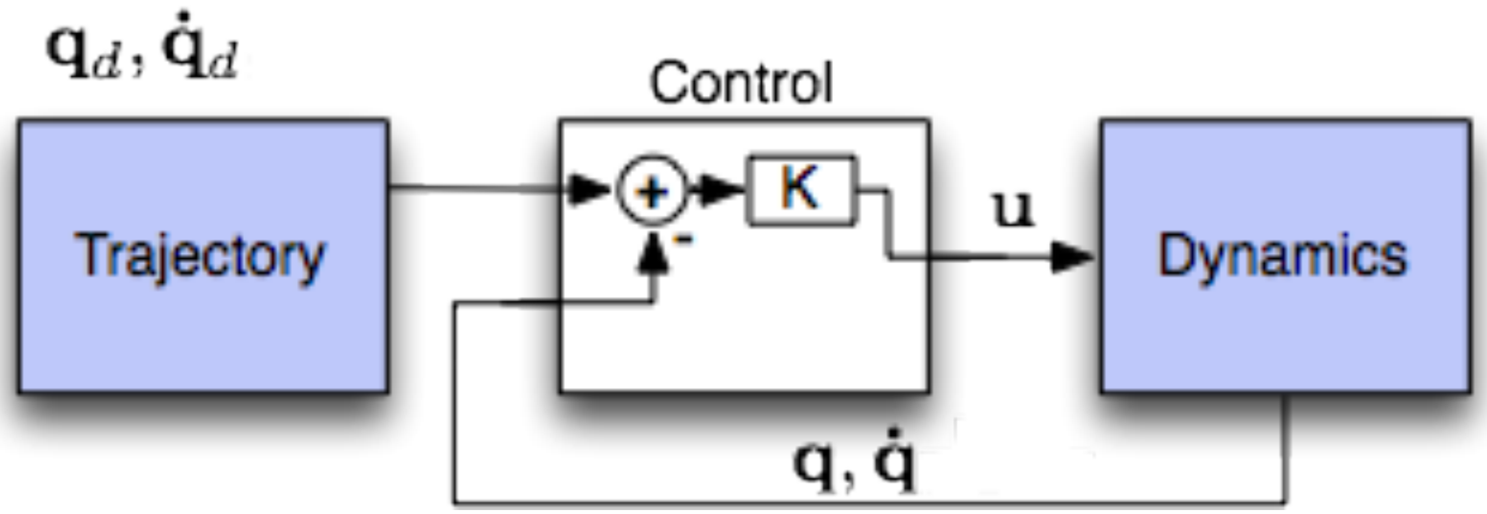


What happens if the sign is messed up?

Check K = -0.2.



Linear Control in Robotics



- P-Controller
- PD-Controller
- PID-Controller

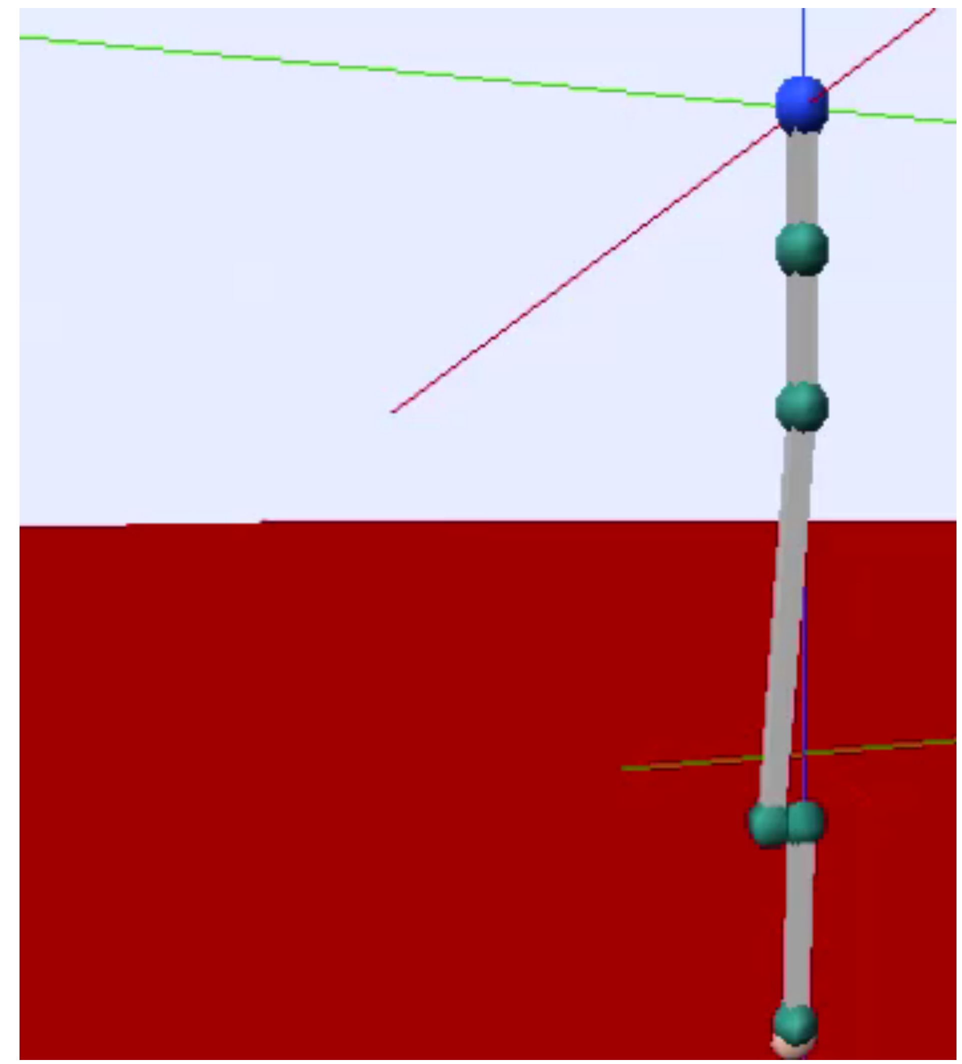
Linear Control: "P-Regler"

P-Controller:

based on position error

$$\boldsymbol{u}_t = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}_t)$$

$$\mathbf{q}_d = \begin{bmatrix} 0 \\ 0.9 \\ 0 \\ 0.9 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \dot{\mathbf{q}}_d = 0$$



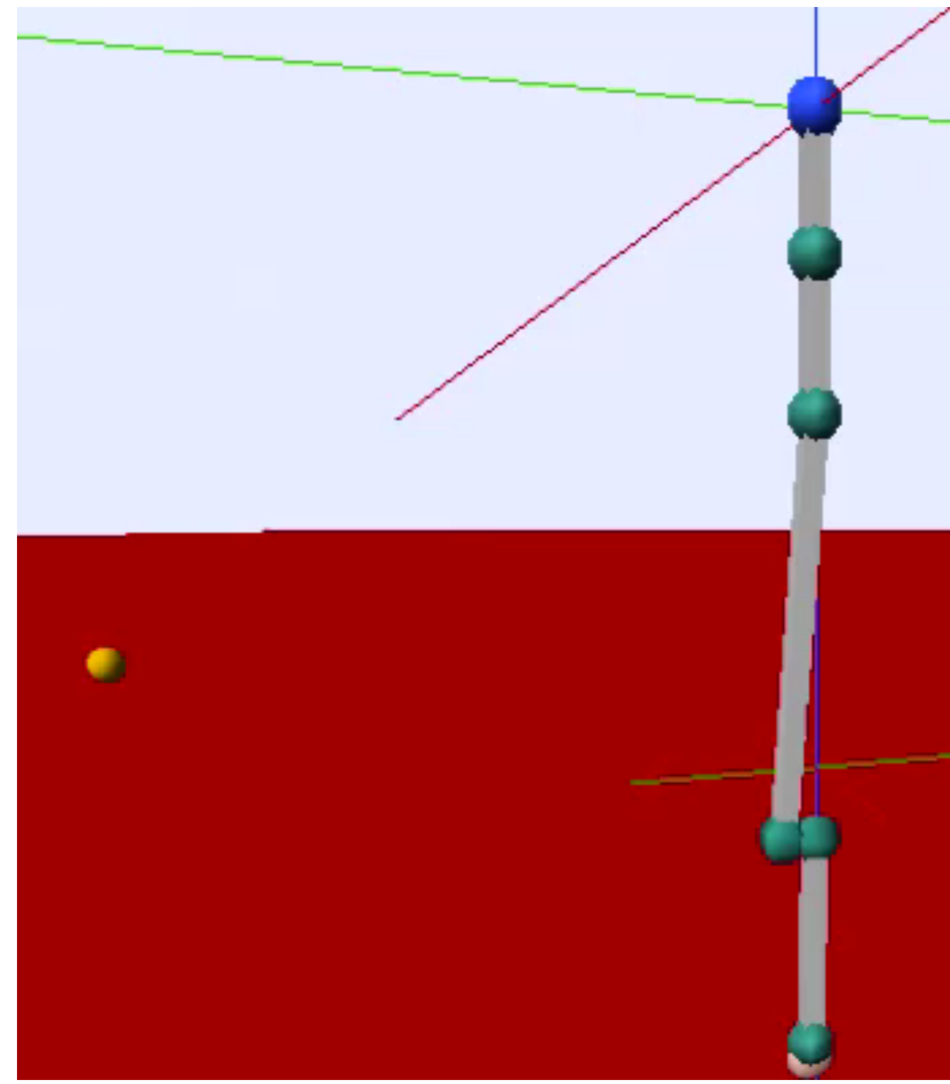
What happens for this control law? → Oscillations

Linear Control: “PD-Regler”

PD-Controller:

based on position and velocity errors

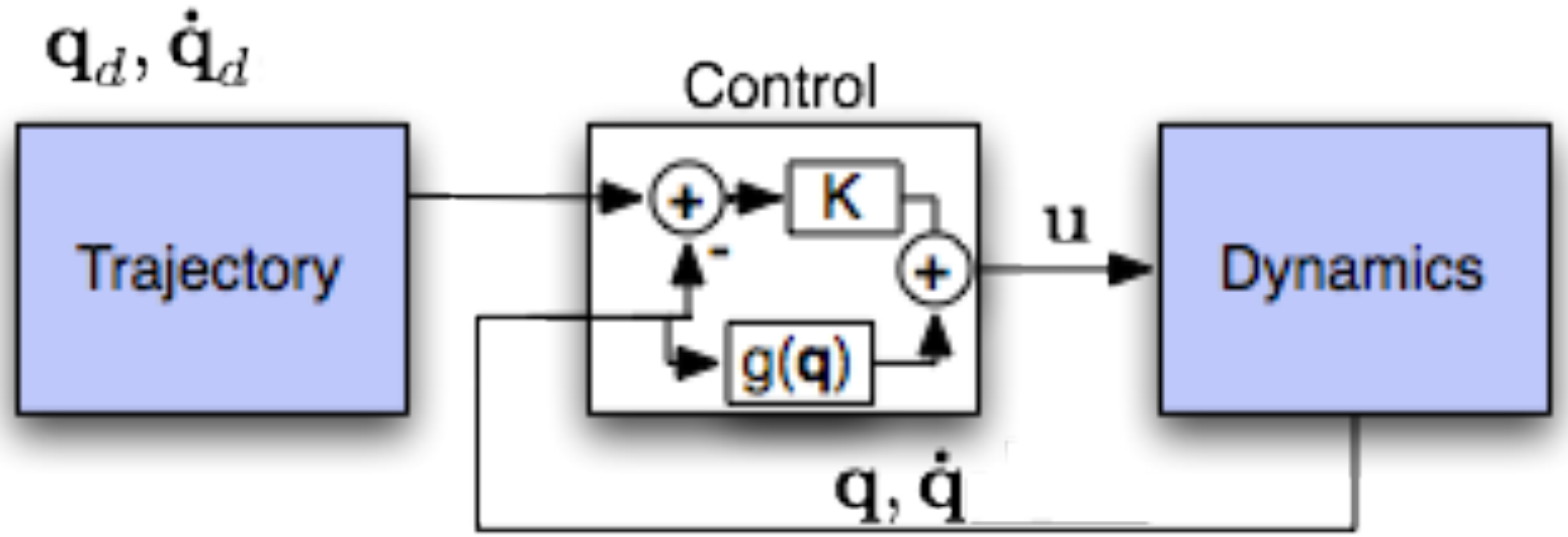
$$\boldsymbol{u}_t = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}_t) + \boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}_t)$$



What happens for this control law?

Less oscillations, but can not reach set-point

Linear PD Control with Gravity Compensation



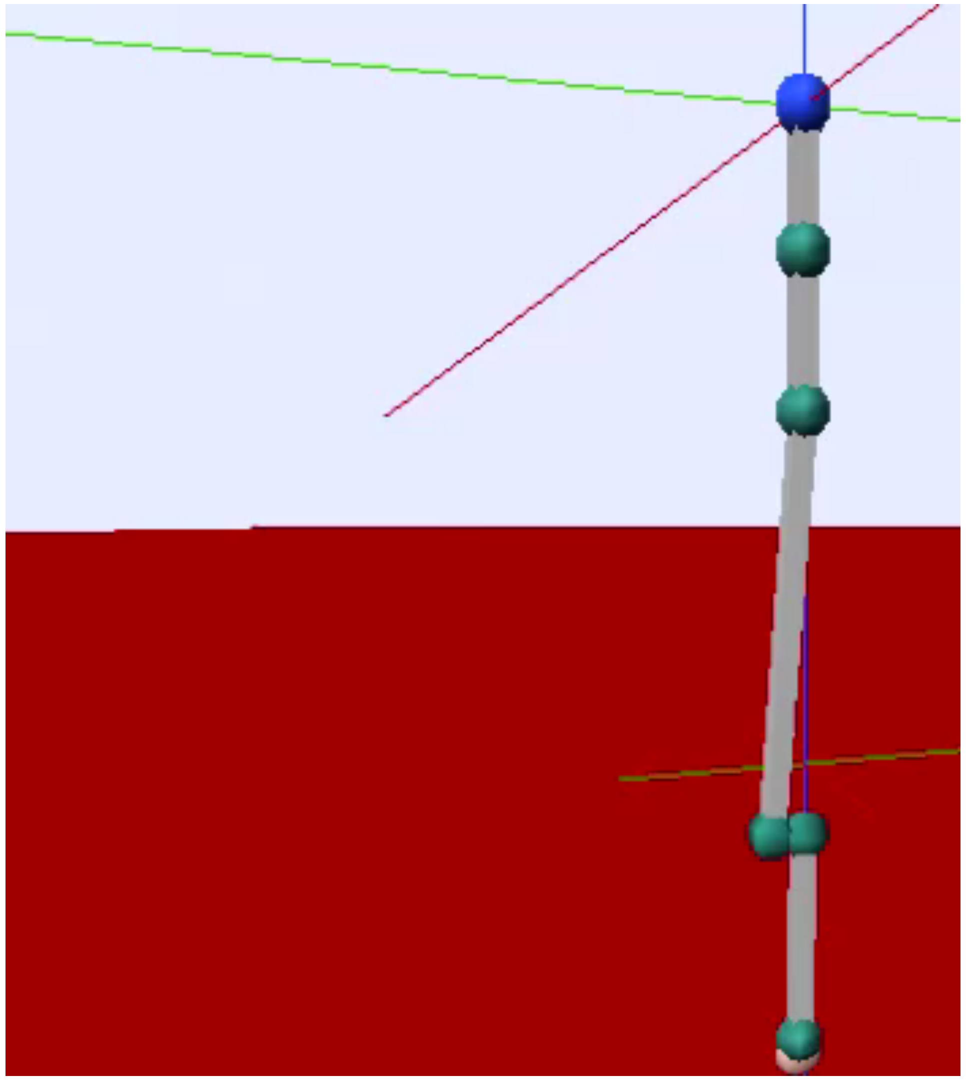
- To reach the set-point, we must compensate for gravity
- Most industrial robots employ this approach

Linear PD Control with Gravity Compensation

PD-Controller with gravity compensation

$$
\begin{aligned}
\boldsymbol{u}_t &= \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}_t) + \boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}_t) \\
&\quad + \boldsymbol{g}(\boldsymbol{q})
\end{aligned}
$$

➡ Requires a model



Note on PID Control

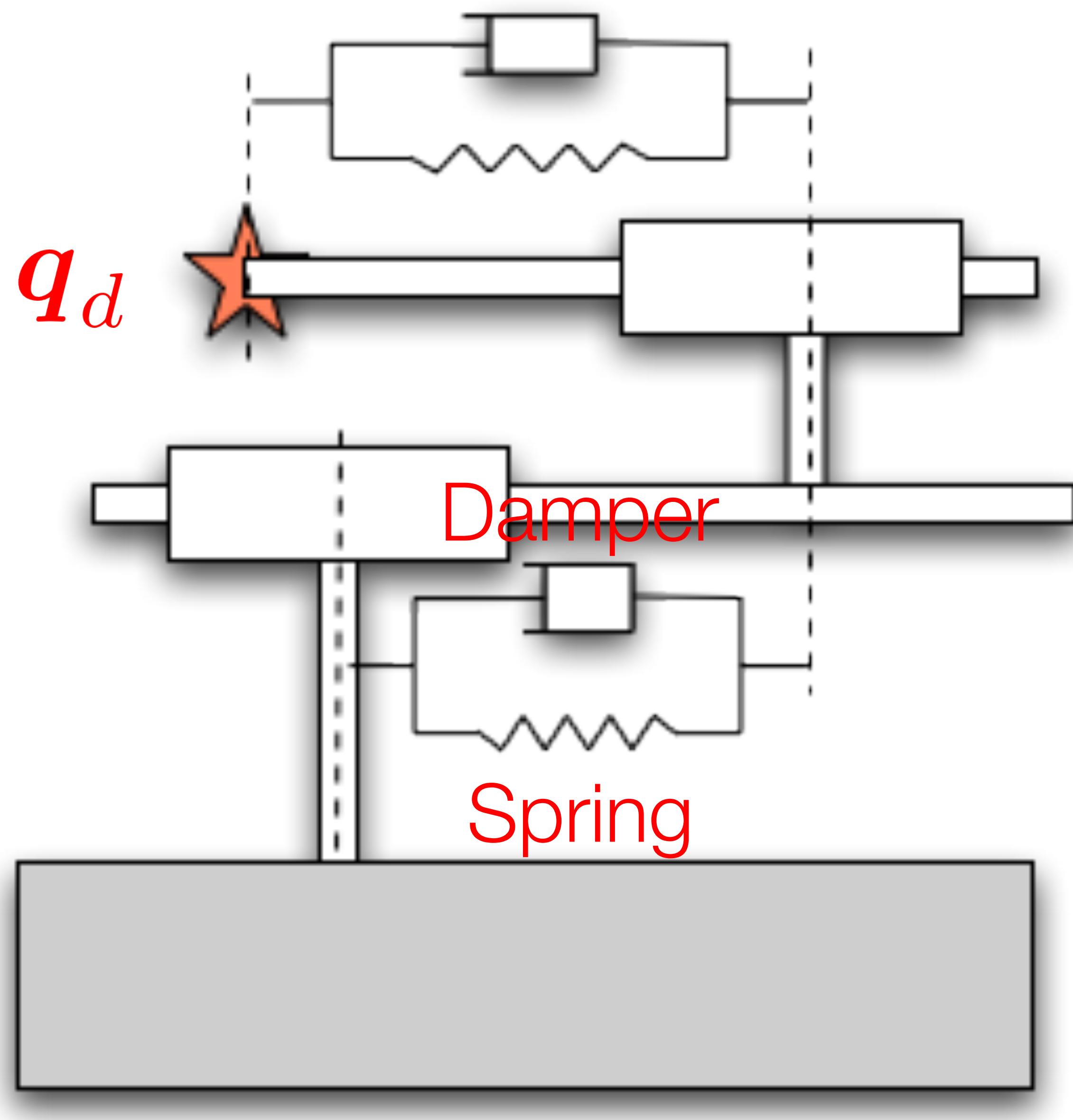
- Alternatively to doing gravity compensation, we could try to estimate the motor command to compensate for the error.
- This can be done by integrating the error

$$\mathbf{u} = \mathbf{K}_P(\mathbf{q}_{\text{des}} - \mathbf{q}) + \mathbf{K}_D(\dot{\mathbf{q}}_{\text{des}} - \dot{\mathbf{q}}) + \mathbf{K}_I \int_{-\infty}^{t} (\mathbf{q}_{\text{des}} - \mathbf{q}) d\tau.$$

- For steady state systems, this can be reasonable (e.g., if our shower thermostat has an offset)
  - good if model is not known!
- For tracking control, it may create havoc and disaster!

Mechanical Equivalent

PD Control is equivalent to adding spring-dampers between the desired values and the actuated robot parts.



$$\boldsymbol{u}_t = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}_t) - \boldsymbol{K}_D\dot{\boldsymbol{q}}_t$$

Ask questions...

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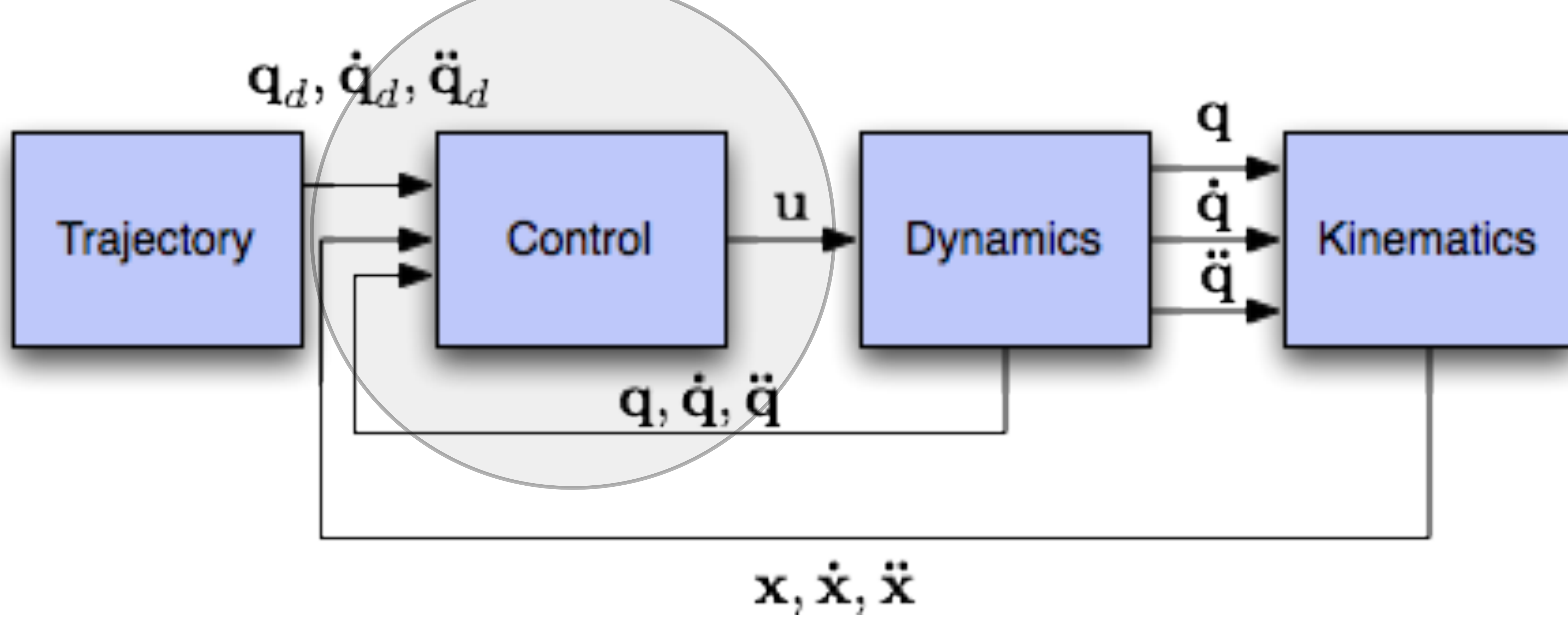
Block Diagram of Complete System

PD with gravity compensation still **can not track a trajectory perfectly**

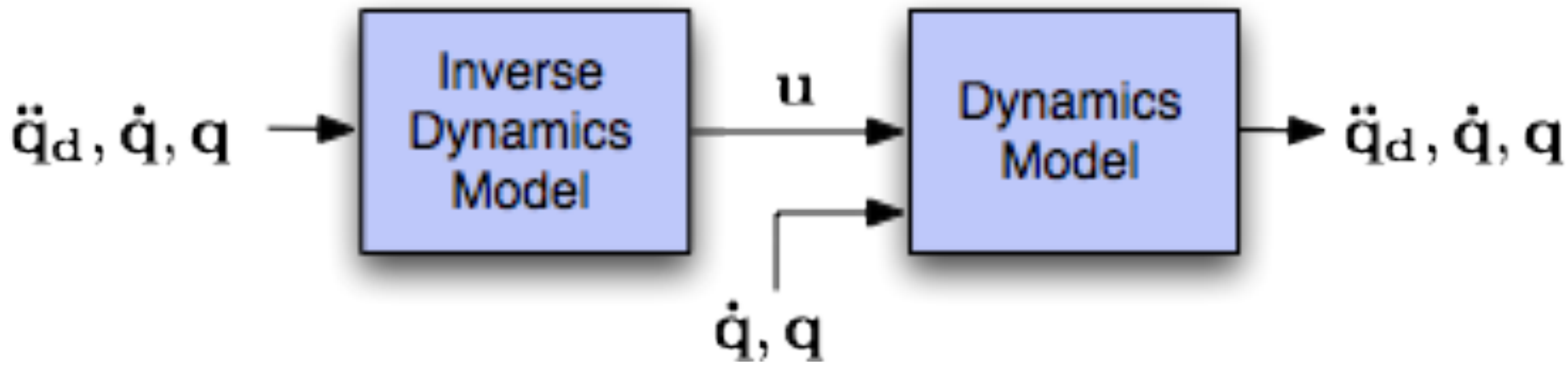
- ➡ We need an error to generate a control signal
- ➡ We do not know which accelerations we produce

Can we do better with a model?

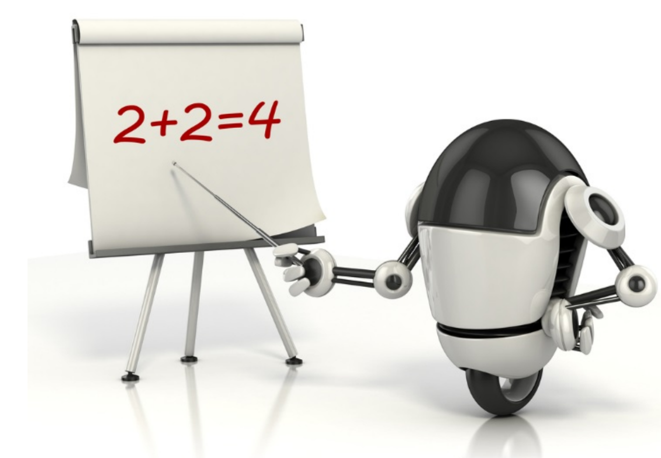


Model-based Control: Key Insight

- Forward and inverse dynamics model have a useful property:



  - Forward Model: $\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{q})(\mathbf{u} - \mathbf{c}(\dot{\mathbf{q}}, \mathbf{q}) - \mathbf{g}(\mathbf{q}))$
  - Inverse Model: $\mathbf{u} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}_{\mathrm{d}} + \mathbf{c}(\dot{\mathbf{q}}, \mathbf{q}) + \mathbf{g}(\mathbf{q})$
- Thus, we set $\ddot{\mathbf{q}} = \ddot{\mathbf{q}}_{\mathrm{d}}$

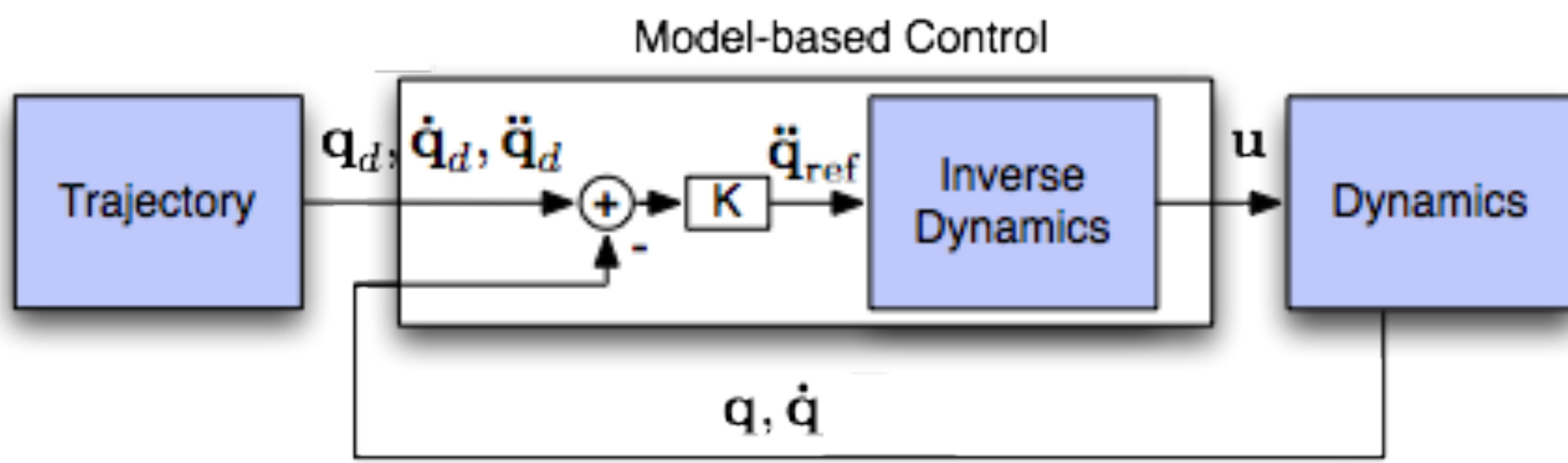


Model-based Feedback Control

- For errors, adapt only reference acceleration

$$\ddot{\mathbf{q}}_{\text{ref}} = \ddot{\mathbf{q}}_{\mathbf{d}} + \mathbf{K}_D(\dot{\mathbf{q}}_{\text{des}} - \dot{\mathbf{q}}) + \mathbf{K}_P(\mathbf{q}_{\text{des}} - \mathbf{q})$$

- ... and insert it into our model $\mathbf{u} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}_{\text{ref}} + \mathbf{c}(\dot{\mathbf{q}}, \mathbf{q}) + \mathbf{g}(\mathbf{q})$
- As $\ddot{q} = \ddot{q}_{\text{ref}}$ the system behaves as linear decoupled system
  - I.e. it is a decoupled double integrator!



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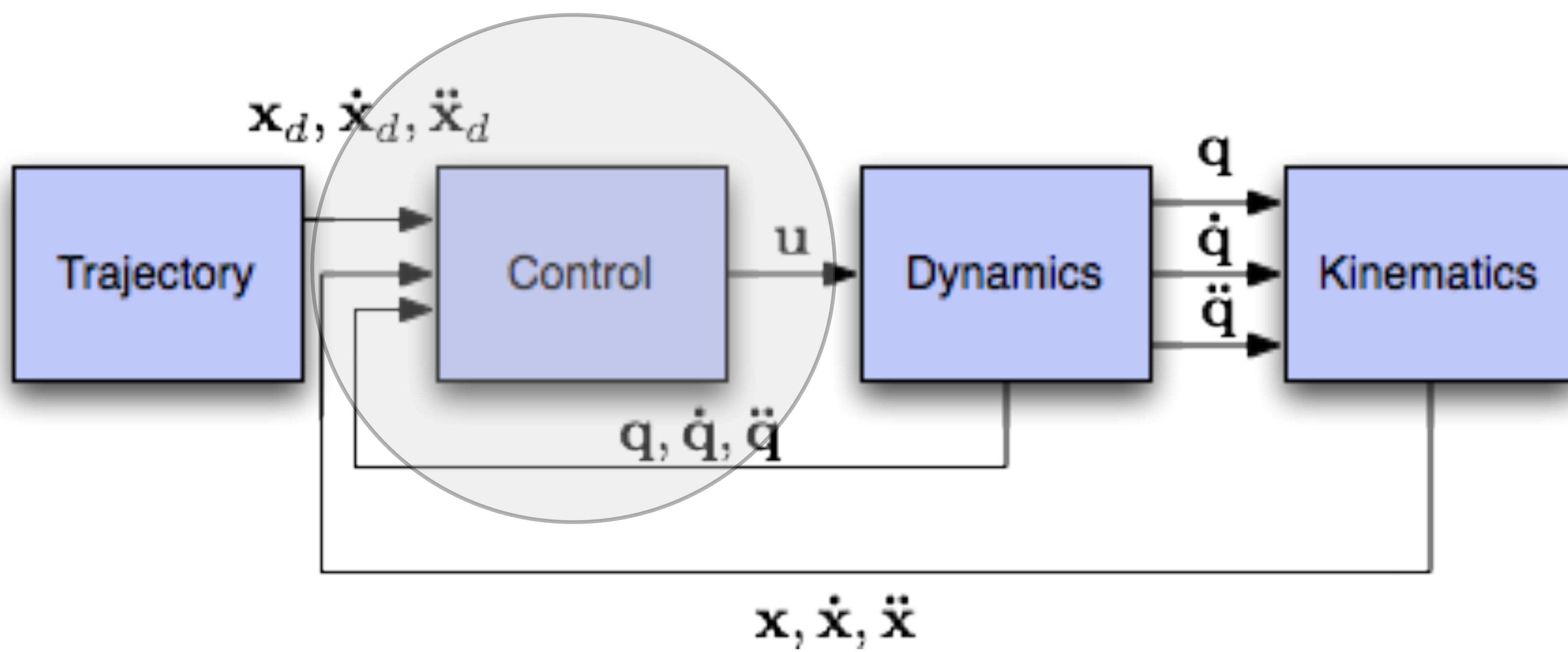
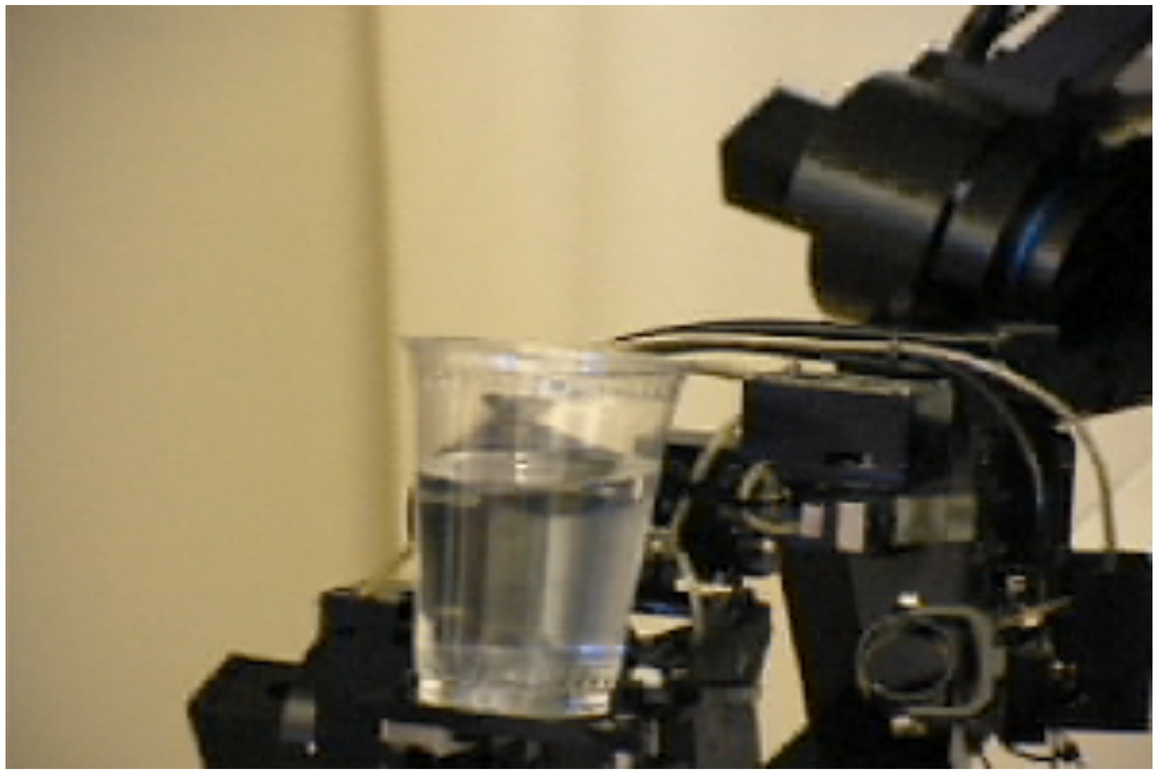
Assume your plan is in a task space...

I.e., we want the end-effector to follow a specific trajectory $\boldsymbol{x}_{1:T}$

- Typically given in Cartesian coordinates
- Eventually also orientation



Inverse Kinematics (IK)



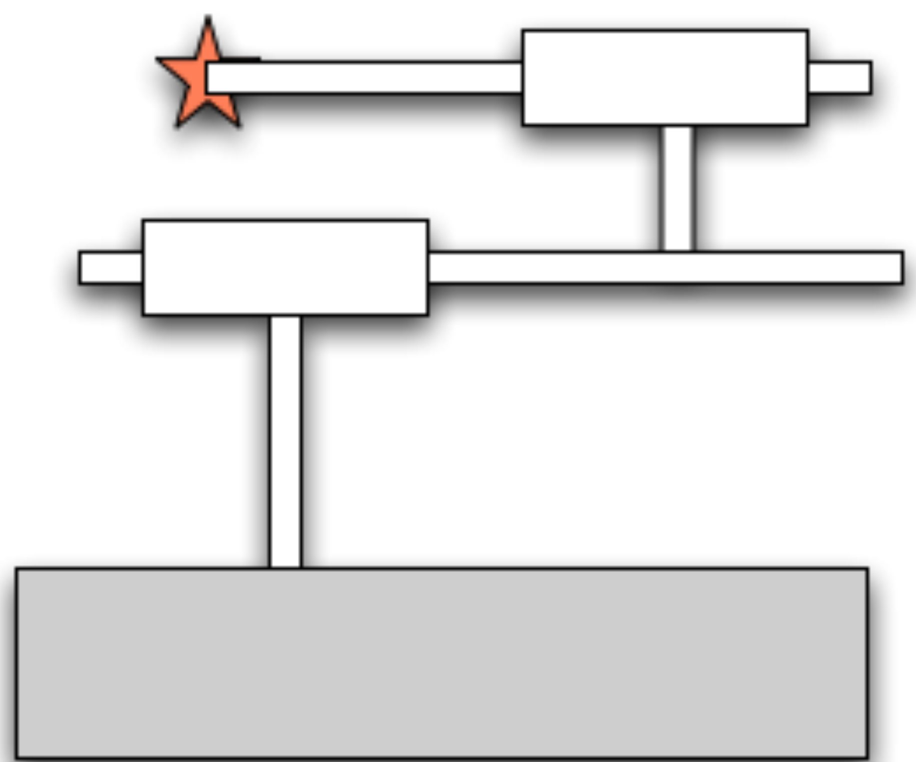
How to move my joints in order to get to a given hand configuration?

If I want my center of gravity in the middle what joint angles do I need?

- **What do we want to have?**
- **Inverse Kinematics:** A mapping from task space to configuration

$$\mathbf{q} = f^{-1}(\mathbf{x})$$

Example 1 - revisited



As $x = q_1 + q_2$

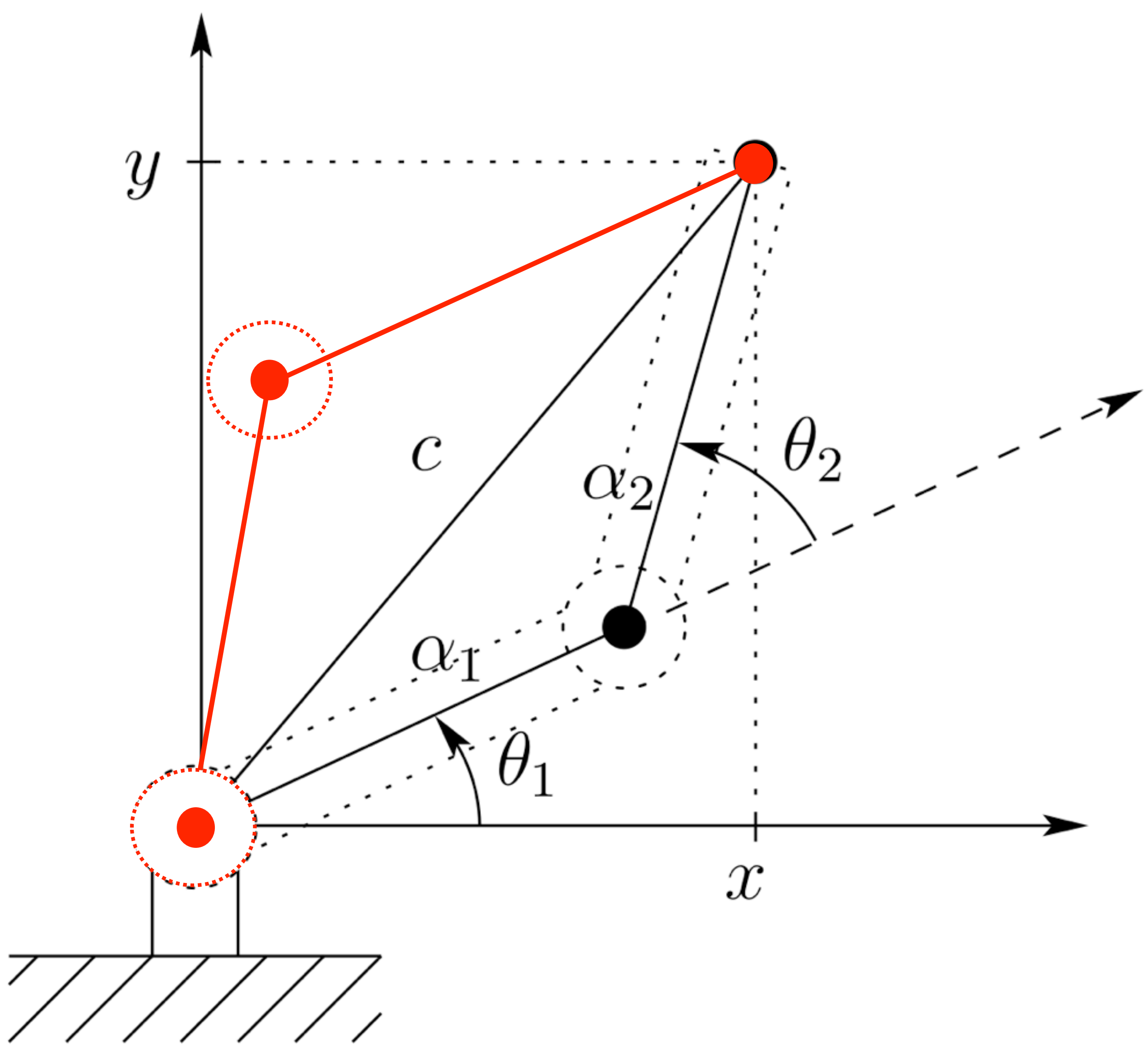
we have

$$\begin{aligned} q_1 &= h \\ q_2 &= x - h \end{aligned}$$

for any $h \in \mathbb{R}$

➡ We have infinitely many solutions!!! Yikes!

Example 2 - revisited



We can solve for θ_1 and θ_2 and get

$$
\begin{aligned}
\theta_2 &= \cos^{-1}\left(\frac{x^2 + y^2 - \alpha_1^2 - \alpha_2^2}{2\alpha_1\alpha 2}\right) \\
\theta_1 &= \tan^{-1}\left(\frac{y}{x}\right) \\
&\quad - \tan^{-1}\left(\frac{\alpha_2 \sin\theta_2}{\alpha_1 + \alpha_2 \cos\theta_2}\right)
\end{aligned}
$$

- BUT: There is more than one solution!
- This is not a function!

Problems with Inverse Kinematics

Multiple solutions even for non-redundant robots (Example 2)

Redundancy results in infinitely many solutions.

➡ Often only numerical solutions are possible!

➡ **Note:** Industrial robots are often built to have invertible kinematics!

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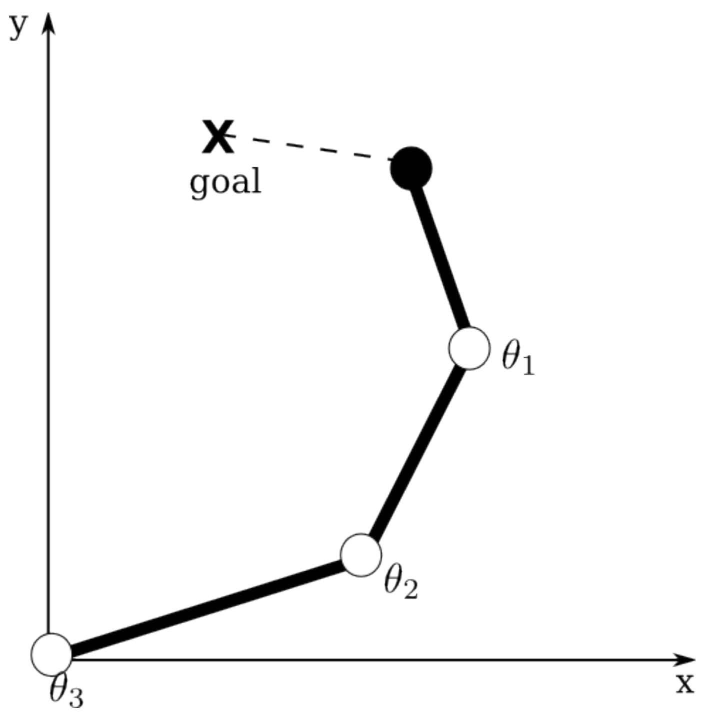
Differential Inverse Kinematics



Inverse kinematics:

$$\boldsymbol{q}_d = f^{-1}(\boldsymbol{x}_d)$$

➡ Not computable as we have an infinite amount of solutions

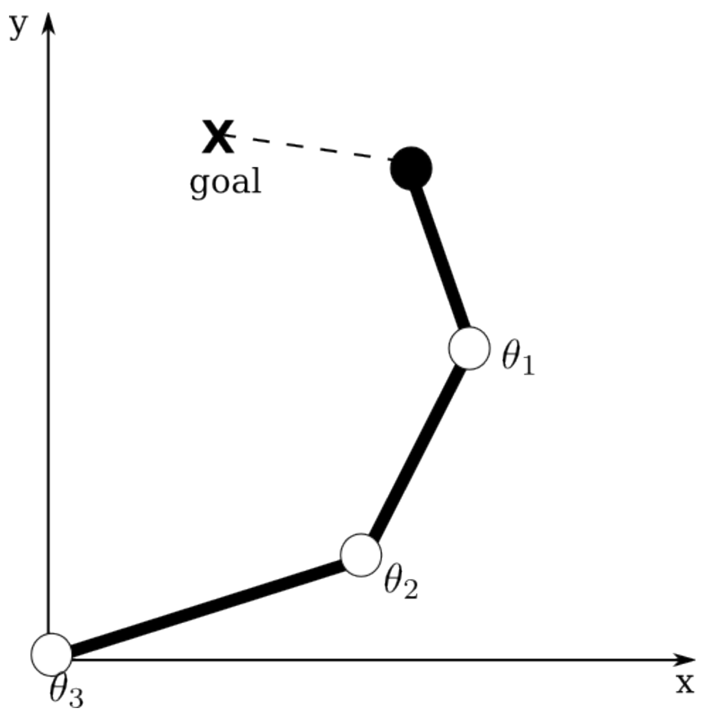
Differential inverse kinematics:

$$\dot{\boldsymbol{q}}_t = \boldsymbol{h}(\boldsymbol{x}_d, \boldsymbol{q}_t)$$

➡ Given current joint positions, compute joint velocities that minimizes the task space error

➡ Computable

Differential Inverse Kinematics



Differential inverse kinematics:

$$\dot{\boldsymbol{q}}_t = \boldsymbol{h}(\boldsymbol{x}_d, \boldsymbol{q}_t)$$

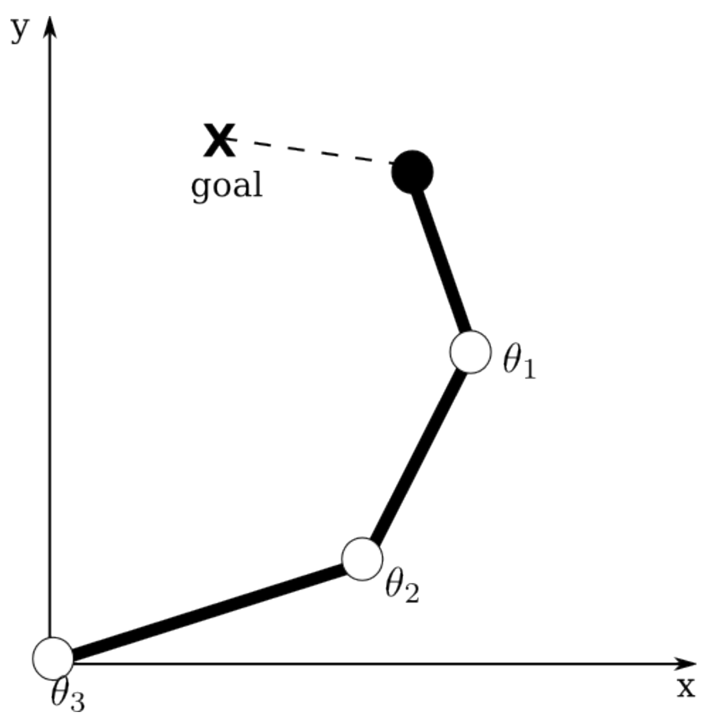
How can we use this for control?

1. Integrate $\dot{\boldsymbol{q}}_t$ and directly use it for joint space control
2. Iterate differential IK algorithm to find $\boldsymbol{q}_d$

   $$\boldsymbol{q}_{k+1} = \boldsymbol{q}_k + h(\boldsymbol{x}_d, \boldsymbol{q}_k)$$

   and plan trajectory to reach $\boldsymbol{q}_d$

Numerical Solution: Jacobian Transpose



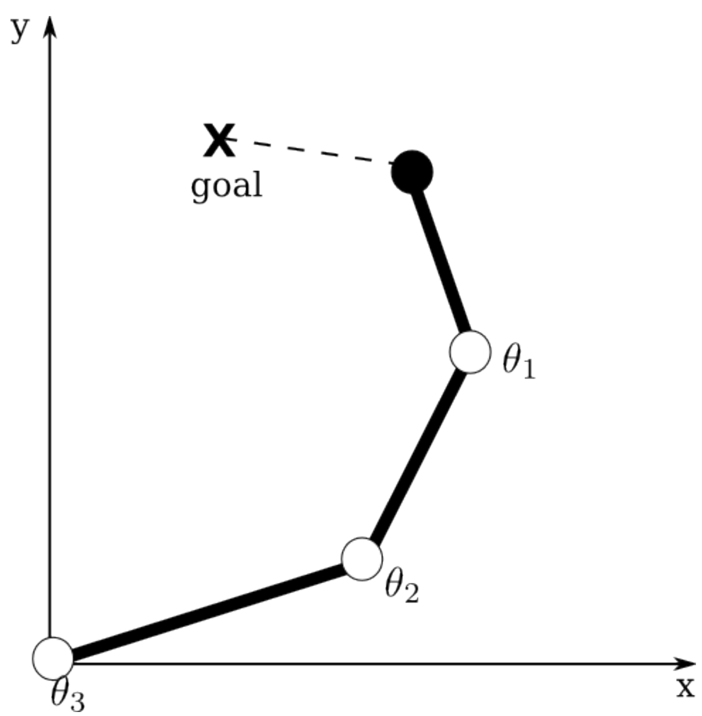
- Minimize the task-space error

$$E = \frac{1}{2}(\mathbf{x} - f(\mathbf{q}))^T(\mathbf{x} - f(\mathbf{q}))$$

- Gradient always points in the direction of steepest ascent

$$\begin{aligned}\frac{dE}{d\boldsymbol{q}} &= -(\boldsymbol{x} - f(\boldsymbol{q}))^T \frac{df(\boldsymbol{q})}{d\boldsymbol{q}} \\ &= -(\boldsymbol{x} - f(\boldsymbol{q}))^T \boldsymbol{J}(\boldsymbol{q})\end{aligned}$$

Jacobian Transpose



Minimize error per gradient descent

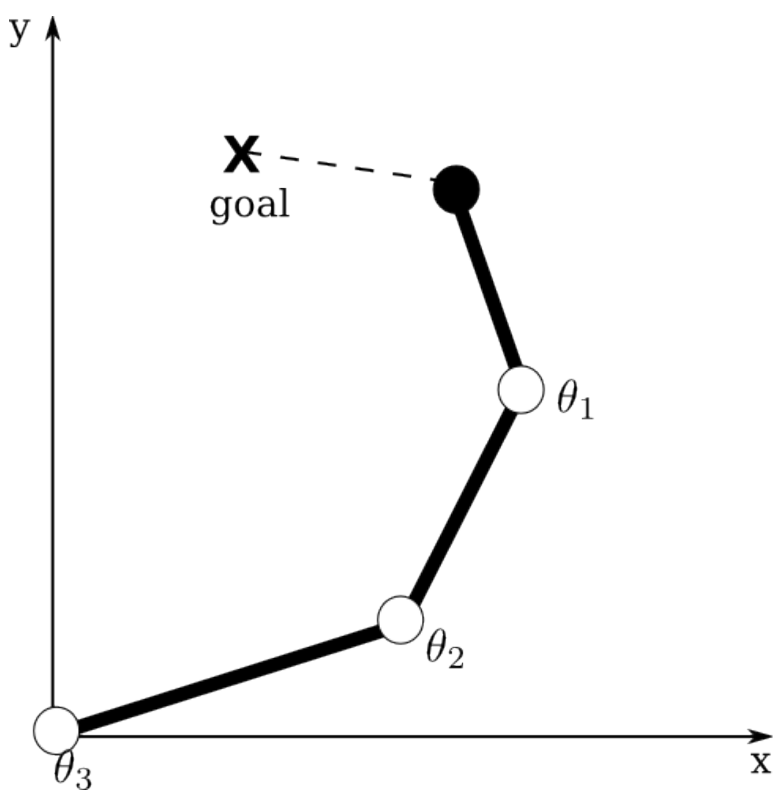
- Follow negative gradient with a certain step size γ

$$\dot{\boldsymbol{q}} = -\gamma \left(\frac{dE}{d\boldsymbol{q}} \right)^T = \gamma \boldsymbol{J}(\boldsymbol{q})^T (\boldsymbol{x} - f(\boldsymbol{q}))$$

$$= \gamma \boldsymbol{J}(\boldsymbol{q})^T \boldsymbol{e}$$

- Known as Jacobian Transpose Inverse Kinematics

Jacobian Pseudo Inverse



- Assume that we are not so far from our solution manifold.
- Take smallest step $\dot{\boldsymbol{q}}$ that has a desired task space velocity

$$\dot{\boldsymbol{x}} = \eta(\boldsymbol{x}_d - f(\boldsymbol{q})) = \eta \boldsymbol{e}$$

- Yields the following optimization problem

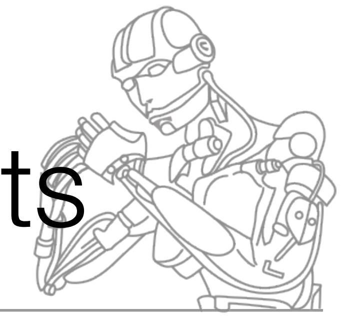
$$\min_{\dot{\boldsymbol{q}}} \; \dot{\boldsymbol{q}}^T \dot{\boldsymbol{q}}, \quad \text{s.t.: } \dot{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- **Solution:** (right) pseudo-inverse

$$\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})^T (\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})^T)^{-1} \dot{\boldsymbol{x}}$$

$$= \eta \boldsymbol{J}(\boldsymbol{q})^{\dagger} \boldsymbol{e}$$

Task-Prioritization with Null-Space Movements



Execute another task $\dot{\boldsymbol{q}}_0$ simultaneously in the "Null-Space"

- For example, "push" robot to a rest-posture

  $$\dot{\boldsymbol{q}}_0 = \boldsymbol{K}_P(\boldsymbol{q}_{\text{rest}} - \boldsymbol{q})$$

- Take step that has smallest distance to "base" task

  $$\min_{\dot{\boldsymbol{q}}} \; (\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_0)^T(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_0), \quad \text{s.t.: } \dot{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- **Solution:** $\dot{\boldsymbol{q}} = \boldsymbol{J}^\dagger \dot{\boldsymbol{x}} + (\boldsymbol{I} - \boldsymbol{J}^\dagger \boldsymbol{J})\dot{\boldsymbol{q}}_0$

  - **Null-Space:** $(\boldsymbol{I} - \boldsymbol{J}^\dagger \boldsymbol{J})$

  - All movements $\dot{\boldsymbol{q}}_{\text{null}}$ that do not contradict the constraint

    $$\dot{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{q})(\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}_{\text{null}}) \text{ or } \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}_{\text{null}} = 0$$

Task-Prioritization with Null-Space Movements

Similarly, we can also use a acceleration formulation

Solution: $\dot{\boldsymbol{q}} = \boldsymbol{J}^{\dagger}(\ddot{\boldsymbol{x}} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}) + (\boldsymbol{I} - \boldsymbol{J}^{\dagger}\boldsymbol{J})\ddot{\boldsymbol{q}}_0$

Problem: However, the inversion in the pseudo-inverse

$\boldsymbol{J}^{\dagger} = \boldsymbol{J}^{T}(\boldsymbol{J}\boldsymbol{J}^{T})^{-1}$ can be problematic

In the case of singularities, $\boldsymbol{J}\boldsymbol{J}^{T}$ can not be inverted!

Damped Pseudo Inverse

Numerically more stable solution:

- Find a tradeoff between minimizing the error and keeping the joint movement small

$$\min_{\dot{\boldsymbol{q}}} \; (\dot{\boldsymbol{x}} - \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}})^T(\dot{\boldsymbol{x}} - \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}) + \lambda \dot{\boldsymbol{q}}^T \dot{\boldsymbol{q}}$$

- Regularization constant λ
- Damped Pseudo Inverse Solution

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T + \lambda \boldsymbol{I})^{-1}\dot{\boldsymbol{x}} = \boldsymbol{J}^{\dagger(\lambda)}\dot{\boldsymbol{x}}$$

- Works much better for singularities

Ask questions...

Q & A?