State Space Modeling

Topics Covered

- Inverted pendulum modeling.
- Introduction to state-space models.
- Model validation.

Prerequisites

- Integration laboratory experiment.
- First Principles Modeling laboratory experiment.
- Pendulum Moment of Inertia laboratory experiment.
- Rotary pendulum module is attached to the QUBE-Servo 2.

1 Background

1.1 Pendulum Model

The rotary pendulum model is shown in Figure 1.1. The rotary arm pivot is attached to the QUBE-Servo 2 system and is actuated. The arm has a length of L_r, a moment of inertia of J_r, and its angle θ increases positively when it rotates counter-clockwise (CCW). The servo (and thus the arm) should turn in the CCW direction when the control voltage is positive ($V_m > 0$).

The pendulum link is connected to the end of the rotary arm. It has a total length of L_p and it center of mass is at $L_p/2$. The moment of inertia about its center of mass is J_p. The inverted pendulum angle α is zero when it is hanging downward and increases positively when rotated CCW.

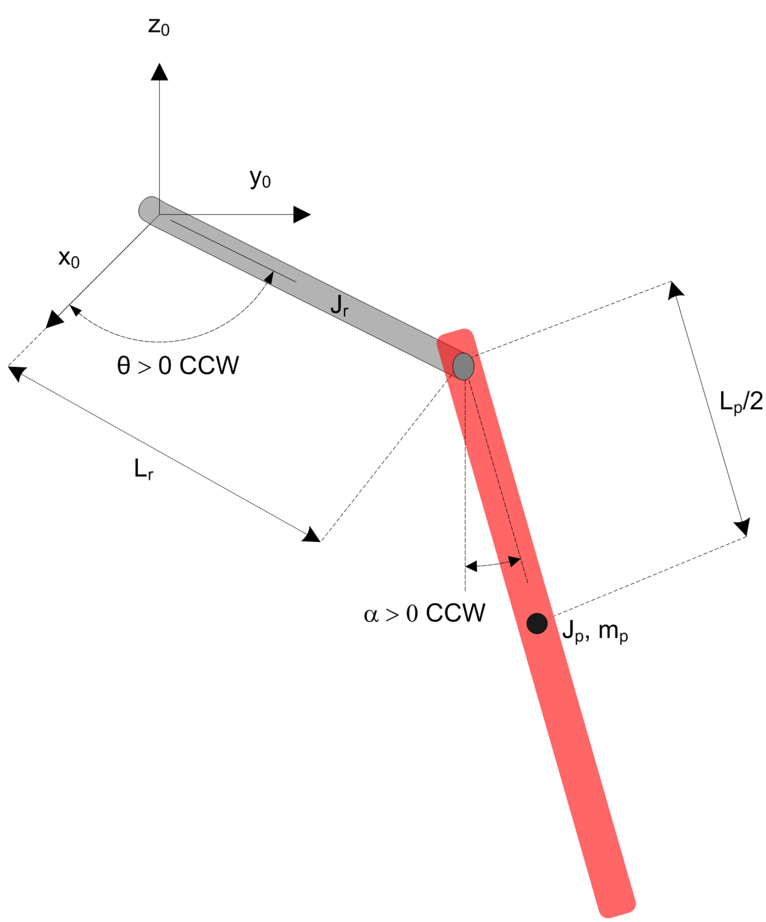

Figure 1.1: Rotary inverted pendulum model

The equations of motion (EOM) for the pendulum system were developed using the Euler-Lagrange method. This systematic method is often used to model complicated systems such as robot manipulators with multiple joints. The total kinetic and potential energy of the system is obtained, then the Lagrangian can be found. A number of derivatives are then computed to yield the EOMs. The complete derivation of the EOM for the pendulum system are presented in the *Rotary Pendulum Modeling Summary* and Maple workbook.

The resultant nonlinear EOM are:

$$\left(m_p L_r^2 + \frac{1}{4} m_p L_p^2 - \frac{1}{4} m_p L_p^2 \cos(\alpha)^2 + J_r\right) \ddot{\theta} - \left(\frac{1}{2} m_p L_p L_r \cos(\alpha)\right) \ddot{\alpha} + \left(\frac{1}{2} m_p L_p^2 \sin(\alpha)\cos(\alpha)\right) \dot{\theta}\dot{\alpha} + \left(\frac{1}{2} m_p L_p L_r \sin(\alpha)\right) \dot{\alpha}^2 = \tau - D_r \dot{\theta} \tag{1.1}$$

$$\frac{1}{2} m_p L_p L_r \cos(\alpha)\ddot{\theta} + \left(J_p + \frac{1}{4} m_p L_p^2\right) \ddot{\alpha} - \frac{1}{4} m_p L_p^2 \cos(\alpha)\sin(\alpha)\dot{\theta}^2 + \frac{1}{2} m_p L_p g \sin(\alpha) = -D_p \dot{\alpha}. \tag{1.2}$$

with an applied torque at the base of the rotary arm generated by the servo motor as described by the equation:

$$\tau = \frac{k_m \left(V_m - k_m \dot{\theta}\right)}{R_m} \tag{1.3}$$

When the nonlinear EOM are linearized about the operating point, the resultant linear EOM for the inverted pendulum are defined as:

$$\left(m_p L_r^2 + J_r\right) \ddot{\theta} - \frac{1}{2} m_p L_p L_r \ddot{\alpha} = \tau - D_r \dot{\theta}. \tag{1.4}$$

and

$$\frac{1}{2} m_p L_p L_r \ddot{\theta} + \left(J_p + \frac{1}{4} m_p L_p^2\right) \ddot{\alpha} + \frac{1}{2} m_p L_p g \alpha = -D_p \dot{\alpha}. \tag{1.5}$$

Solving for the acceleration terms yields:

$$\ddot{\theta} = \frac{1}{J_T} \left(-\left(J_p + \frac{1}{4} m_p L_p^2\right) D_r \dot{\theta} + \frac{1}{2} m_p L_p L_r D_p \dot{\alpha} + \frac{1}{4} m_p^2 L_p^2 L_r g \alpha + \left(J_p + \frac{1}{4} m_p L_p^2\right) \tau\right). \tag{1.6}$$

and

$$\ddot{\alpha} = \frac{1}{J_T} \left(\frac{1}{2} m_p L_p L_r D_r \dot{\theta} - \left(J_r + m_p L_r^2\right) D_p \dot{\alpha} - \frac{1}{2} m_p L_p g \left(J_r + m_p L_r^2\right) \alpha - \frac{1}{2} m_p L_p L_r \tau\right). \tag{1.7}$$

where

$$J_T = J_p m_p L_r^2 + J_r J_p + \frac{1}{4} J_r m_p L_p^2. \tag{1.8}$$

1.1.1 Linear State-Space Model

The linear state-space equations are

$$\dot{x} = Ax + Bu \tag{1.9}$$

and

$$y = Cx + Du \tag{1.10}$$

where x is the state, u is the control input, A, B, C and D are state-space matrices. For the rotary pendulum system, the state and output are defined

$$x = \begin{bmatrix} \theta & \alpha & \dot{\theta} & \dot{\alpha} \end{bmatrix}^T \tag{1.11}$$

and

$$y = \begin{bmatrix} \theta & \alpha \end{bmatrix}^T. \tag{1.12}$$

Swing-Up Control

Topics Covered

- Energy control.
- Nonlinear control.
- Control switching logic.

Prerequisites

- Filtering laboratory experiment.
- Rotary Pendulum Modeling laboratory experiment.
- Rotary pendulum module is attached to the QUBE-Servo 2.

1 Background

1.1 Energy Control

In theory, if the arm angle is kept constant and the pendulum is given an initial perturbation, the pendulum will keep on swinging with constant amplitude. The idea of energy control is based on the preservation of energy in ideal systems: The sum of kinetic and potential energy is constant. However, friction will be damping the oscillation in practice and the overall system energy will not be constant. It is possible to capture the loss of energy with respect to the pivot acceleration, which in turn can be used to find a controller to swing up the pendulum.

The dynamics of the pendulum can be redefined in terms of the pivot acceleration u as

$$J_p\ddot{\alpha} + \frac{1}{2}M_p g L_p \sin\alpha = \frac{1}{2}M_p L_p u \cos\alpha. \tag{1.1}$$

Here, u is the linear acceleration of the pendulum.

The potential energy of the pendulum is

$$E_p = \frac{1}{2}M_p g L_p \left(1 - \cos\alpha\right),$$

and the kinetic energy is

$$E_k = \frac{1}{2}J_p\dot{\alpha}^2.$$

The pendulum angle α and the lengths of the pendulum are illustrated in the free body diagram in Figure 1.1.

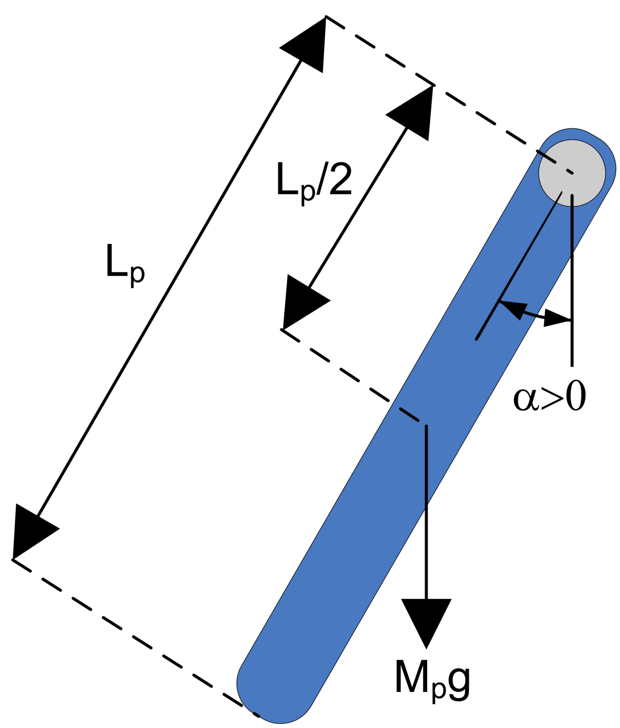

Figure 1.1: Free-body diagram of pendulum

The potential energy is zero when the pendulum is at rest at $\alpha = 0$ and equals $M_p g L_p$ when the pendulum is upright at $\alpha = \pm\pi$. The sum of the potential and kinetic energy of the pendulum is

$$E = \frac{1}{2}J_p\dot{\alpha}^2 + \frac{1}{2}M_p g L_p \left(1 - \cos\alpha\right). \tag{1.2}$$

Differentiating Equation 1.2 yields

$$\dot{E} = \dot{\alpha}\left(J_p\ddot{\alpha} + \frac{1}{2}M_p g L_p \sin\alpha\right). \tag{1.3}$$

Recalling Equation 1.1 and rearranging terms as

$$J_p\ddot{\alpha} = -\frac{1}{2}M_p g L_p \sin\alpha + \frac{1}{2}M_p u L_p \cos\alpha$$

eventually yields

$$\dot{E} = \frac{1}{2}M_p u L_p \dot{\alpha}\cos\alpha.$$

Since the acceleration of the pivot is proportional to current driving the arm motor and thus also proportional to the drive voltage, it is possible to control the energy of the pendulum with the proportional control law

$$u = (E_r - E)\dot{\alpha}\cos\alpha. \tag{1.4}$$

By setting the reference energy to the pendulum potential energy ($E_r = E_p$), the control law will swing the link to its upright position. Notice that the control law is nonlinear because the proportional gain depends on the cosine of the pendulum angle α. Further, the control changes sign when $\dot{\alpha}$ changes sign and when the angle is ± 90 degrees.

For the system energy to change quickly, the magnitude of the control signal must be large. As a result the following swing-up controller is implemented in the controller as

$$u = \text{sat}_{u_{max}}\left(\mu(E_r - E)\text{sign}(\dot{\alpha}\cos\alpha)\right) \tag{1.5}$$

where μ is a tunable control gain and the $\text{sat}_{u_{max}}$ function saturates the control signal at the maximum acceleration of the pendulum pivot, u_{max}. The expression $\text{sign}(\dot{\alpha}\cos\alpha)$ is used to enable faster control switching.

1.2 Hybrid Swing-Up Control

The energy swing-up control in Equation 1.4 (or Equation 1.5) can be combined with the balancing control law from the Balance Control Lab to obtain a control law that swings up the pendulum and then balances it.

Similarly as described in the Rotary Pendulum Modeling laboratory experiment, the balance control is to be enabled when the pendulum is within ± 20 degrees. When it is not enabled, the swing-up control is engaged. Thus the switching can be described mathematically by:

$$u = \begin{cases} u_{bal} & \text{if } |\alpha| - \pi \leq 20^\circ \\ u_{swing\_up} & \text{otherwise} \end{cases} \tag{1.6}$$